

On the proper meaning of the curvature tensor and its general framework

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Abstract

We make evident a curvature tensor for every vector subbundle of an arbitrary manifold tangent bundle which reduces to the curvature tensor of an Ehresmann connection in the case of the horizontal subbundle of the tangent bundle to the total space of the nonlinear fiber bundle on which the connection is defined. Then the classical theorem of Frobenius would characterize the complete integrability of a vector subbundle of the tangent bundle by a zero curvature tensor in the sense of our definition here. A basic tool is a result about the curvature tensor of the natural lift of the vector subbundle to a manifold of maps with values in the base of that subbundle. Another is a localization property for a Lie algebra of vector fields over this manifold of maps. These allow to prove an additive formula for the curvature tensors of two supplementary subbundles. The main result consists in identifying a natural linear parallel transport on a supplementary vector subbundle along any tangent path to the vector subbundle under study, which is the right generalization of a linear connection parallel transport on a vector bundle along the projection in the base of that path. Then we derive the differential equation of the quotient of respective parallel transport operators induced by two different supplementary subbundles to the subbundle in question in terms of its curvature. Using this we obtain the equation of the infinitesimal variation of tangent paths to a vector subbundle, defined by its curvature, that appears as the root for the Jacobi equation of the infinitesimal variation of geodesics.

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1 Introduction. The curvature tensor of a vector subbundle of the tangent bundle

This paper is intended to fill a gap, due to a remarkable blind hens' phenomenon, regarding the notion of curvature. After the important work of Ehresmann, this missing step forward needed sixty four years! I became aware of the right definition of the curvature tensor some seven years ago, but I could not believe it is not known: in the paper [3] I took this definition for granted and I sent for it to the book [7] - where it does not appear! Only after looking through the beautiful book of Sternberg [6] I decided first to write a half page paper devoted only to this definition, in fact the content of Theorem 1 here. This approach was suggested by the double formulation (and proof) of the Frobenius theorem from the book [4] of Narasimhan. Indeed, there is used the formula (18) from here to pass from one formulation to the other. And precisely this formula allows to introduce our *definition of the curvature tensor for every smooth vector subbundle of any manifold tangent bundle*. In fact, for $H_p \subseteq T_p M$ smooth vector subbundle, the curvature tensor at $p \in M$ is a linear operator $C_p^H \in \text{Hom}(H_p \wedge H_p, T_p M/H_p)$. Then the theorem

of Frobenius asserts the equivalence between the complete integrability of such a vector subbundle and the equality to zero, at every point of the manifold, of its curvature tensor, in the sense of our definition here. The local expression of the curvature tensor shows that the Christoffel symbols appear even in this general case and that their meaning comes from the local representation of the respective vector subbundle of TM using charts on M and on a Grassmann manifold.

Next, in section §2, it is shown that for any $p_0 \in M$, $H_0 \subseteq T_{p_0}M$ vector subspace and $C_0 \in \text{Hom}(H_0 \wedge H_0, T_{p_0}M/H_0)$ there is a locally defined, around p_0 , vector subbundle H of TM such that $H_{p_0} = H_0$ and $C_{p_0}^H = C_0$. Taking into account the integrability property of the subbundle H in the most degenerate case, when $C_p^H = 0$ for all $p \in M$, we point out in Proposition 4, dual properties of non-integrability of H corresponding to the two cases when the linear operator C_p^H is non-degenerate, hence injective or surjective, respectively. Having in view to study tangent paths to H in M and variations of them through tangent paths to H alike, we consider in section §3 the manifolds $C^\infty(D, M)$, for D a compact domain, meant to be of dimension 1 or 2 respectively. If $H \rightarrow M$ is the vector subbundle of TM , there is a natural lift of it to a vector bundle $C^\infty(D, H) \rightarrow C^\infty(D, M)$, which appears to be a vector subbundle of $TC^\infty(D, M)$, if we consider the (formal) isomorphism $TC^\infty(D, M) \xrightarrow{\sim} C^\infty(D, TM)$. The result of §3, Theorem 2, essential in all the paper, is a simple and natural formula for the curvature of $C^\infty(D, H)$, as a vector subbundle of $TC^\infty(D, M)$, in terms of the curvature of H .

The section §4 is devoted to a Lie subalgebra of vector fields over $C^\infty(D, M)$ which is isomorphic to the set of global sections of a sheaf of Lie algebras over $D \times M$. The localization property of this Lie algebra structure is used in §5 to prove a certain identity (Theorem 4) that entails the formula expressing the sum of curvatures of two supplementary subbundles by such a Lie bracket (Theorem 5). But the formula for the Lie bracket of this sheaf sections (Theorem 3) will play also a role in section §6.

This section §6 is the core of the whole paper. We show that for every tangent to the subbundle H path γ and for every supplementary to H vector subbundle K there exists a natural linear parallel transport of the fibers of K along the path γ . It is the right generalization of the case when M is the total space of a vector bundle, over B say, and γ is the zero lift of an arbitrary path in B . Then γ is tangent to the horizontal subbundle H defining a linear connection on M . Next, the linear fibers of M , over B , can be identified with the tangent to them in zero, giving thus the supplementary subbundle K along γ . And, in this case, the parallel transport of the fibers of M along the original path in the base B , given by the linear connection, coincides with the parallel transport of the fibers of K along γ . If M is the total space of a nonlinear fiber bundle on which is defined an Ehresmann connection by H and K is the tangent to the nonlinear fibers, then the linear parallel transport operators of K along γ , found by us, appear to be the tangent operators, in the points of the tangent to H path γ , to the diffeomorphisms of nonlinear fibers given by the Ehresmann connection parallel transport along the projection of γ in the base (Theorem 8).

Coming back to our general framework, if we consider the linear parallel transport op-

erators of the fibers of TM/H defined by two supplementary subbundles K^1 and K^2 to H , then there is an interesting equation of evolution of their quotient, along γ , in terms of the curvature of H (Theorem 10). This equation is the consequence of Theorem 9, where we find the equation of the infinitesimal variation of tangent paths to H using a supplementary subbundle K to H . Then in section §7 we prove the reciprocal of this statement and show that the equation, defined by the curvature of H , is independent of K (Theorems 11 and 12). Finally, we show in Theorem 13 that this equation is the root for the equation of Jacobi of infinitesimal variation of geodesics.

2 The definition of the curvature tensor and its expression in local coordinates. The two nondegenerate cases and some integrability questions

Let M be a C^∞ (finite dimensional) manifold and $H_p \subseteq T_p M, p \in M$, be a C^∞ vector subbundle of its tangent bundle. For a diffeomorphism $\varphi : M \longrightarrow N$ we denote $\varphi_* H$ the vector subbundle of TN :

$$(\varphi_* H)_{\varphi(p)} = T_p \varphi \cdot H_p, \quad p \in M. \quad (1)$$

For a Banach vector space E we consider

$$\Psi^E : E \times E \longrightarrow TE, \quad \Psi^E(v, w) = \frac{d}{dt}(v + tw)|_{t=0} \in T_v E, \quad (2)$$

which is a bijection and $\Psi^E(x, \cdot) : E \longrightarrow T_x E$ is an isomorphism $\forall x \in E$. Then, if M is a Banach manifold, E a Banach vector space and $f : M \longrightarrow E$ is a C^1 - function, we denote

$$d_p f : T_p M \longrightarrow E, \quad d_p f = \Psi^E(f(p), \cdot)^{-1} \cdot T_p f \quad (3)$$

its differential at $p \in M$. And if $\chi : U \longrightarrow E, \quad U = \overset{\circ}{U} \subseteq M$, is a local chart on M , we consider the subspaces of E

$$\widetilde{(\chi_* H)}_{\chi(p)} = d_p \chi \cdot H_p, \quad p \in U. \quad (4)$$

Using a chart in the neighbourhood of $p_0 \in M$ we may construct a local trivialization of H around p_0 in the following way: let $V := \widetilde{(\chi_* H)}_{\chi(p_0)}$ and W such that $V \dot{+} W = E$, where $\dot{+}$ stands for the interior direct sum. Then a natural isomorphism allows to consider a modified chart

$$\chi : U \longrightarrow V \times W \quad (5)$$

such that $d_{p_0}\chi \cdot H_{p_0} = V \times \{0_W\}$. The smoothness of H is equivalent to the smoothness of the map $p \mapsto d_p\chi \cdot H_p$ taking values in the Grassmann manifold of subspaces of $V \times W$ isomorphic with V . In this way we find $\Gamma(x, y) \in \text{Hom}(V, W)$ depending smoothly on $(x, y) \in V \times W$ in a neighbourhood of $(x_0, y_0) = \chi(p_0)$ such that $\Gamma(x_0, y_0) = 0_{\text{Hom}(V, W)}$ and

$$\widetilde{(\chi_* H)}_{(x, y)} = \text{graph } \Gamma(x, y) = \{(v, \Gamma(x, y)v) \mid v \in V\}. \quad (6)$$

In that neighbourhood we consider the isomorphism

$$\Lambda_{(x, y)}^V : V \xrightarrow{\sim} \widetilde{(\chi_* H)}_{(x, y)}, \quad \Lambda_{(x, y)}^V v = (v, \Gamma(x, y)v), \quad (7)$$

and finally the trivialization of H

$$\vartheta_p : H_p \xrightarrow{\sim} V, \quad \vartheta_p = (\Lambda_{\chi(p)}^V)^{-1} \cdot d_p\chi|_{H_p}, \quad p \in U. \quad (8)$$

We will call a chart (5) where H is represented by (6) an *adapted chart* for the vector subbundle H . If $H_p^\perp \subseteq T_p^*M$ denotes the orthogonal with respect to the duality $\{T_p M, T_p^* M\}$ then H^\perp becomes a C^∞ vector subbundle of T^*M . In the case of a H -adapted chart we denote

$$\Lambda_{(x, y)}^W : W \xrightarrow{\sim} E / \text{graph } \Gamma(x, y), \quad \Lambda_{(x, y)}^W w = (0_V, w) + \text{graph } \Gamma(x, y), \quad (9)$$

where $E = V \times W$, and then its transposed establishes an isomorphism

$$(\Lambda_{(x, y)}^W)^* : (\text{graph } \Gamma(x, y))^\perp \longrightarrow W^*. \quad (10)$$

On the other hand $((d_p\chi)^{-1})^* : T_p^*M \longrightarrow E^*$ is an isomorphism between H_p^\perp and $(d_p\chi \cdot H_p)^\perp$, so that

$$\vartheta_p^\perp : H_p^\perp \xrightarrow{\sim} W^*, \quad \vartheta_p^\perp = (\Lambda_{\chi(p)}^W)^* \cdot ((d_p\chi)^{-1})^*|_{H_p^\perp} \quad (11)$$

gives a trivialization of H^\perp in the same neighbourhood. There, a section $X \in C^\infty\Gamma(H)$ is represented by $f = \vartheta \circ X \circ \chi^{-1} \in C^\infty(U, V)$ such that

$$d_{\chi^{-1}(x, y)}\chi \cdot X_{\chi^{-1}(x, y)} = (f(x, y), \Gamma(x, y)f(x, y)) \quad (12)$$

and a section $\alpha \in C^\infty\Gamma(H^\perp)$ by $\varphi = \vartheta_p^\perp \circ \alpha \circ \chi^{-1} \in C^\infty(U, W^*)$ for which

$$\langle ((d_{\chi^{-1}(x, y)}\chi)^*)^{-1}\alpha_{\chi^{-1}(x, y)}, (v, w) \rangle = \langle \varphi(x, y), w - \Gamma(x, y)v \rangle. \quad (13)$$

Our starting point is the following elementary

Theorem 1 . For $X, Y \in C^\infty\Gamma(H)$ and $\alpha \in C^\infty\Gamma(H^\perp)$ the following equality holds $\forall p \in M$:

$$d_p\alpha(X_p, Y_p) = - \langle \alpha_p, [X, Y]_p \rangle. \quad (14)$$

It follows that for $p_0 \in M$ fixed it is well defined a trilinear map

$$\tau_{p_0} : H_{p_0} \times H_{p_0} \times H_{p_0}^\perp \longrightarrow \mathbf{R} \quad (15)$$

by

$$\tau_{p_0}(u, v, \phi) = \langle \phi, [X, Y]_{p_0} \rangle = -d_{p_0}\alpha(u, v), \quad (16)$$

where $X, Y \in C^\infty\Gamma(H)$, $\alpha \in C^\infty\Gamma(H^\perp)$ are arbitrary with

$$X_{p_0} = u, Y_{p_0} = v, \alpha_{p_0} = \phi. \quad (17)$$

Proof. The equality (14) follows from the general identity

$$d_p\alpha(X_p, Y_p) = \langle d_p\langle \alpha, Y \rangle, X_p \rangle - \langle d_p\langle \alpha, X \rangle, Y_p \rangle - \langle \alpha_p, [X, Y]_p \rangle, \quad (18)$$

for $X, Y \in C^\infty\Gamma(TM)$, $\alpha \in C^\infty\Gamma(T^*M)$. Then in (16) the second term shows that τ_{p_0} depends only on ϕ while the third term shows that τ_{p_0} depends only on u and v . On the other hand, the representations (12) and (13) allow to construct sections for H and H^\perp with prescribed value in a given point. Using this procedure we can also verify that τ_{p_0} is trilinear ■

We consider then the following

Definition. If H is a vector subbundle of TM we define its *curvature tensor* in $p_0 \in M$ as the operator

$$C_{p_0}^H \in \text{Hom}(H_{p_0} \wedge H_{p_0}, T_{p_0}M/H_{p_0}), \quad (19)$$

$$C_{p_0}^H(u \wedge v) := P_{p_0}^H[X, Y]_{p_0}, \quad (20)$$

if

$$P_{p_0}^H : T_{p_0}M \longrightarrow T_{p_0}M/H_{p_0} \quad (21)$$

denotes the canonical projection and $X, Y \in C^\infty\Gamma(H)$ are chosen so that $X_{p_0} = u$, $Y_{p_0} = v$ ■

Then the classical *Frobenius theorem* may be phrased as stating that H is *completely integrable if and only if* $C_p^H = 0, \forall p \in M$ (see Narasimhan [4]).

We look now for a local representation of the curvature tensor corresponding to the local form (6) of the vector subbundle H . For any diffeomorphism $\varphi : M \longrightarrow N$ we have also the canonical isomorphism

$$T_p\varphi/H_p : T_pM/H_p \xrightarrow{\sim} T_{\varphi(p)}N/(\varphi_*H)_{\varphi(p)} \quad (22)$$

(see (1)). If C^{φ_*H} is the curvature tensor of the subbundle φ_*H of TN , we have

$$T_p\varphi/H_p \cdot C_p^H(X_p \wedge Y_p) = C_{\varphi(p)}^{\varphi_*H}(T_p\varphi \cdot X_p \wedge T_p\varphi \cdot Y_p), \quad (23)$$

for $X_p, Y_p \in H_p$, $p \in M$. If $\chi : U \longrightarrow E$, $U = \overset{\circ}{U} \subseteq M$, is a local chart on M , we denote (see (4))

$$d_p\chi/H_p : T_pM/H_p \longrightarrow E/(\widetilde{\chi_*H})_{\chi(p)} \quad (24)$$

and define $\widetilde{C_e^{\chi_* H}} \in \text{Hom}(\widetilde{(\chi_* H)_e} \wedge \widetilde{(\chi_* H)_e}, E / \widetilde{(\chi_* H)_e})$, for $e \in E$, by

$$d_p \chi / H_p \cdot C_p^H(X_p \wedge Y_p) = \widetilde{C_{\chi(p)}^{\chi_* H}}(d_p \chi \cdot X_p \wedge d_p \chi \cdot Y_p). \quad (25)$$

For $X \in C^\infty \Gamma(TM)$ we have $\chi_* X = (\chi^{-1})^* X \in C^\infty \Gamma(TE)$ defined as usual

$$(\chi_* X)_e = T_{\chi^{-1}(e)} \chi \cdot X_{\chi^{-1}(e)} \quad (26)$$

and for $X \in C^\infty \Gamma(TE)$ we denote

$$\tilde{X} = p_2 \circ (\Psi^E)^{-1} \circ X, \quad \tilde{X} : E \longrightarrow E, \quad (27)$$

(where $p_2 : E \times E \longrightarrow E$ is canonical), such that

$$X_e = \Psi^E(e, \tilde{X}(e)), \quad e \in E. \quad (28)$$

From (26) and (27) we get

$$\widetilde{\chi_* X}(e) = d_{\chi^{-1}(e)} \chi \cdot X_{\chi^{-1}(e)}. \quad (29)$$

And if we define for $F, G \in C^\infty(E, E)$

$$[F, G](e) = G'(e) \cdot F(e) - F'(e) \cdot G(e), \quad (30)$$

for $X, Y \in C^\infty \Gamma(TE)$ we have

$$[\widetilde{X}, \widetilde{Y}] = [\tilde{X}, \tilde{Y}]. \quad (31)$$

Since $(\chi^{-1})^* [X, Y] = [(\chi^{-1})^* X, (\chi^{-1})^* Y]$, we have

$$\chi_* [\widetilde{X}, \widetilde{Y}] = [\widetilde{\chi_* X}, \widetilde{\chi_* Y}] \quad (32)$$

for $X, Y \in C^\infty \Gamma(TM)$.

In order to distinguish the type of application to different vectors we use here the notations

$$\langle \frac{\partial F}{\partial x}(x, y); f \rangle = \frac{d}{dt} F(x + tf, y)|_{t=0}, \quad \langle \frac{\partial F}{\partial y}(x, y); h \rangle = \frac{d}{dt} F(x, y + th)|_{t=0}, \quad (33)$$

for $F : O \longrightarrow Z$, Z vector space and $O = \mathring{O} \subseteq V \times W$. In the case $Z = \text{Hom}(V, W)$

$\langle \frac{\partial F}{\partial x}(x, y); f \rangle g \in W$ has a clear meaning for $f, g \in V$; also $\langle \frac{\partial F}{\partial y}(x, y); h \rangle g \in W$, for $h \in W, g \in V$.

Coming back to (32) we get for

$$\widetilde{\chi_* X}(x, y) = (f(x, y), \Gamma(x, y) f(x, y)), \quad \widetilde{\chi_* Y}(x, y) = (g(x, y), \Gamma(x, y) g(x, y)) \quad (34)$$

$$\begin{aligned}
[\widetilde{\chi_* X}, \widetilde{\chi_* Y}](x, y) = & \left(\langle \frac{\partial g}{\partial x}; f \rangle + \langle \frac{\partial g}{\partial y}; \Gamma f \rangle - \langle \frac{\partial f}{\partial x}; g \rangle - \langle \frac{\partial f}{\partial y}; \Gamma g \rangle, \right. \\
& \langle \frac{\partial \Gamma}{\partial x}; f \rangle g + \Gamma \langle \frac{\partial g}{\partial x}; f \rangle + \langle \frac{\partial \Gamma}{\partial y}; \Gamma f \rangle g + \Gamma \langle \frac{\partial g}{\partial y}; \Gamma f \rangle - \langle \frac{\partial \Gamma}{\partial x}; g \rangle f - \\
& \left. - \Gamma \langle \frac{\partial f}{\partial x}; g \rangle - \langle \frac{\partial \Gamma}{\partial y}; \Gamma g \rangle f - \Gamma \langle \frac{\partial f}{\partial y}; \Gamma g \rangle \right).
\end{aligned}$$

Let us consider

$$P_e^{\widetilde{\chi_* H}} : E \longrightarrow E/(\widetilde{\chi_* H})_e, \quad e \in E, \quad (35)$$

canonical and remark that (see (24))

$$d_p \chi / H_p \cdot P_p^H = P_{\chi(p)}^{\widetilde{\chi_* H}} \cdot d_p \chi, \quad (36)$$

$p \in U$. Then

$$\begin{aligned}
P_{(x,y)}^{\widetilde{\chi_* H}}[\widetilde{\chi_* X}, \widetilde{\chi_* Y}](x, y) = \\
= P_{(x,y)}^{\widetilde{\chi_* H}}(0, \langle \frac{\partial \Gamma}{\partial x}; f \rangle g - \langle \frac{\partial \Gamma}{\partial x}; g \rangle f + \langle \frac{\partial \Gamma}{\partial y}; \Gamma f \rangle g - \langle \frac{\partial \Gamma}{\partial y}; \Gamma g \rangle f),
\end{aligned}$$

or (see(9))

$$\begin{aligned}
P_{(x,y)}^{\widetilde{\chi_* H}}[\widetilde{\chi_* X}, \widetilde{\chi_* Y}](x, y) = \\
= \Lambda_{(x,y)}^W \left(\langle \frac{\partial \Gamma}{\partial x}; f \rangle g - \langle \frac{\partial \Gamma}{\partial x}; g \rangle f + \langle \frac{\partial \Gamma}{\partial y}; \Gamma f \rangle g - \langle \frac{\partial \Gamma}{\partial y}; \Gamma g \rangle f \right) \quad (37)
\end{aligned}$$

Let us consider

$$\widehat{C_{(x,y)}^{\chi_* H}} \in \text{Hom}(V \wedge V, W) \quad (38)$$

defined by

$$\begin{aligned}
\widehat{C_{(x,y)}^{\chi_* H}}(f \wedge g) = & \langle \frac{\partial \Gamma}{\partial x}(x, y); f \rangle g - \langle \frac{\partial \Gamma}{\partial x}(x, y); g \rangle f + \\
& + \langle \frac{\partial \Gamma}{\partial y}(x, y); \Gamma(x, y) f \rangle g - \langle \frac{\partial \Gamma}{\partial y}(x, y); \Gamma(x, y) g \rangle f. \quad (39)
\end{aligned}$$

As $(\widetilde{\chi_* [X, Y]})_{\chi(p)} = d_p \chi \cdot [X, Y]_p$, from (32), (37) and (39) we get

$$d_p \chi / H_p \cdot C_p^H((d_p \chi)^{-1} \Lambda_{\chi(p)}^V f \wedge (d_p \chi)^{-1} \Lambda_{\chi(p)}^V g) = \Lambda_{\chi(p)}^W \cdot \widehat{C_{\chi(p)}^{\chi_* H}}(f \wedge g), \quad (40)$$

for $f, g \in V$. We say that (39) is the *local representation of the curvature tensor* in an adapted chart. We recall that here H is defined by Γ according to (6).

The formula (39) shows the proper and general meaning of the *Christoffel symbols*, as

defined by Γ from (6), in virtue of the role of Γ in the expression of the curvature tensor (see, for instance, Kobayashi+Nomizu, vol I [2], Sternberg [6]): they simply give locally the subbundle whose curvature is computed.

We examine now the structure of the curvature tensor at a fixed point.

Proposition 1 . *For $p_0 \in M$ fixed, the operator $C_{p_0}^H$ may take any value from $\text{Hom}(H_{p_0} \wedge H_{p_0}, T_{p_0}M/H_{p_0})$, when H_p varies with p around p_0 and H_{p_0} is prescribed.*

Proof. We consider

$$M = V \times W \quad (41)$$

and $H_{(x,y)}$ given by (6) and $C_{(x,y)}^H$ by (39). We take next $C \in \text{Hom}(V \wedge V, W)$ arbitrary and define H by

$$\Gamma(x, y)v = \frac{1}{2}C(x \wedge v), \quad x, v \in V, \quad y \in W. \quad (42)$$

Then $\langle \frac{\partial \Gamma}{\partial x}(x, y); u \rangle v = \frac{1}{2}C(u \wedge v)$, $\frac{\partial \Gamma}{\partial y}(x, y) = 0$, wherefrom, using also (39), for this H :

$$C_{(x,y)}^H = C, \quad \forall (x, y) \in V \times W \quad (43)$$

■

The following fact comes straight from the definition of the curvature tensor.

Proposition 2 . *Let $H_p \subseteq K_p \subseteq T_pM$ be two smooth vector subbundles and let*

$$j_p : H_p \longrightarrow K_p, \quad q_p : T_pM/H_p \longrightarrow T_pM/K_p \quad (44)$$

be the canonical maps. Then $\forall p \in M$

$$C_p^K(j_p u \wedge j_p v) = q_p C_p^H(u \wedge v), \quad \forall u, v \in H_p. \quad (45)$$

■

Next we try to identify in suitable properties of the curvature tensor the existence of any of two types of integral submanifolds.

Proposition 3 .i) *If $N \subseteq M$ is a submanifold with*

$$T_p N \subseteq H_p, \quad \forall p \in N, \quad (46)$$

then

$$C_p^H(u \wedge v) = 0, \quad \forall p \in N, \quad \forall u, v \in T_p N. \quad (47)$$

ii) *If $N \subseteq M$ is a submanifold with*

$$T_p N \supseteq H_p, \quad \forall p \in N, \quad (48)$$

then

$$C_p^H(u \wedge v) \in T_p N/H_p, \quad \forall p \in N, \quad \forall u, v \in H_p \quad (49)$$

(as $T_p N/H_p \subseteq T_p M/H_p$).

Proof. It is easy to see that for each submanifold N and each $p \in N$ there exists a neighbourhood around p in M and a completely integrable subbundle $K \subseteq TM$ defined on it such that

$$K_p = T_p N, \quad \forall p \in N. \quad (50)$$

- i) In this case $K_p \subseteq H_p$ and $C_p^K = 0$; then from (45) we infer (47).
- ii) In this case $K_p \supseteq H_p$ and $C_p^K = 0$; then (45) gives $q_p C_p^H(u \wedge v) = 0, \quad \forall p \in N, \quad u, v \in H_p$. But $\ker q_p = K_p/H_p = T_p N/H_p$ (see (44) and (50)) ■

We will say that the curvature tensor is *nondegenerate* at p_0 if the operator $C_{p_0}^H \in \text{Hom}(H_{p_0} \wedge H_{p_0}, T_{p_0} M/H_{p_0})$ has maximal rank. Hence there are two cases, depending on $m = \dim H_{p_0}$ and $n = \dim T_{p_0} M$: in the case that $\binom{m}{2} \leq n - m$, $C_{p_0}^H$ should be injective and in the case that $\binom{m}{2} \geq n - m$, $C_{p_0}^H$ should be surjective. Of course, if the curvature tensor is nondegenerate at a certain point, then it is nondegenerate at each point from a neighbourhood of it.

Proposition 4 .i) If $C_{p_0}^H$ is injective then there does not exist a submanifold $N \subseteq M$, $N \ni p_0$, such that

$$T_p N \subseteq H_p, \quad \forall p \in N, \quad \dim N \geq 2. \quad (51)$$

ii) If $C_{p_0}^H$ is surjective then there does not exist a submanifold $N \subseteq M$, $N \ni p_0$, such that

$$T_p N \supseteq H_p, \quad \forall p \in N, \quad \text{codim } N \geq 1. \quad (52)$$

Proof.i) From (47), Proposition 3 i) and the injectivity it follows that $T_p N \wedge T_p N \subseteq \ker C_p^H = \{0\}$, $\forall p \in N$ and in a neighbourhood of p_0 , and then $\dim N \leq 1$.
ii) Using Proposition 3 ii) and the surjectivity we get $T_p N/H_p = T_p M/H_p$, hence $T_p N = T_p M$ and $\text{codim } N = 0$ ■

But the previous Proposition can be supplemented with the counterexamples from the next

Proposition 5 .i) In the case that C_p^H is injective $\forall p \in M$ it may exist a submanifold $N \subseteq M$ with

$$T_p N \supseteq H_p, \quad \forall p \in N, \quad \text{codim } N \geq 1. \quad (53)$$

More precisely, this is possible in the case of (41), (42) with C injective and

$$\dim(V \times W) = 4, \quad \dim V = \dim H = 2, \quad (54)$$

when we can find N of

$$\dim N = 3. \quad (55)$$

ii) In the case that C_p^H is surjective $\forall p \in M$ it may exist a submanifold $N \subseteq M$ with

$$T_p N \subseteq H_p, \quad \forall p \in N, \quad \dim N \geq 2. \quad (56)$$

Again, this can be realized for (41), (42) with C surjective,

$$\dim(V \times W) = 4, \quad \dim V = \dim H = 3, \quad (57)$$

with a certain N of

$$\dim N = 2. \quad (58)$$

Proof.i) We take bases such that $V = \mathbf{R}e_1 + \mathbf{R}e_2$, $W = \mathbf{R}e_3 + \mathbf{R}e_4$ and define C by

$$C(e_1 \wedge e_2) = e_3. \quad (59)$$

Then from (42) we have $\text{im } \Gamma(x', x'') = \mathbf{R}e_3$, $\forall x' \in V$, $x' \neq 0$, $\forall x'' \in W$ and $H_{(x', x'')} = \{v \oplus \Gamma(x', x'')v \mid v \in V\} \subset \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3$. Therefore the hypersurfaces $N_\lambda \subset V \times W$,

$$N_\lambda = \{x_4 = \lambda\}, \quad (60)$$

$\lambda \in \mathbf{R}$, with $T_{(x', x'')}N_\lambda = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3$, have the desired properties (53), (55).

ii) We take $V = \mathbf{R}e_1 + \mathbf{R}e_2 + \mathbf{R}e_3$, $W = \mathbf{R}e_4$, $C : V \wedge V \longrightarrow W$ by

$$C(e_1 \wedge e_2) = C(e_2 \wedge e_3) = 0, \quad C(e_1 \wedge e_3) = e_4. \quad (61)$$

Let us consider the surfaces

$$N_\lambda = \{x_3 = 0, x_4 = \lambda\}, \quad (62)$$

$\lambda \in \mathbf{R}$. For $x = (x', x'') \in N_\lambda$, $x' \in V$, $x'' \in W$, we have $x' \in \mathbf{R}e_1 + \mathbf{R}e_2$. On the other hand $T_{(x', x'')}N_\lambda = \mathbf{R}e_1 + \mathbf{R}e_2$ and if $z \in T_{(x', x'')}N_\lambda$ then $C(x' \wedge z) = 0$ wherefrom $z = z + \frac{1}{2}C(x' \wedge z) = z + \Gamma(x', x'')z \in H_{(x', x'')}$; thus (56) holds ■

3 The lift of the curvature tensor to a manifold of maps with values in the base of the subbundle

The formula that is proved here involves the Lie bracket of vector fields on the infinite dimensional manifold $C^\infty(D, M)$, where D is an arbitrary *compact* manifold (possibly with boundary) and M is the manifold where the vector subbundles of TM are considered. The special form of the vector fields that appear in it allows to work with a certain Lie algebra that does not use a differentiable structure on $C^\infty(D, M)$, being enough to compute on the manifolds $C^k(D, M)$ modelled on Banach spaces. We refer to R. Palais [5] for the facts concerning these Banach manifolds that we use in the sequel. The suitable framework is that of C^∞ nonlinear fiber bundles $\pi : \eta \longrightarrow D$ and of the respective Banach manifolds $C^k\Gamma(\eta)$ of C^k sections.

The main tool in order to introduce a structure of C^∞ manifold on $C^k\Gamma(\eta)$ is the existence, for each $\sigma_0 \in C^0\Gamma(\eta)$, of a *vector bundle neighbourhood* of it, that is an open subset $E \subseteq \eta$

with $E \supseteq \sigma_0(D)$ such that the fiber bundle $\pi|_E : E \longrightarrow D$ be endowed with a compatible structure of C^∞ vector bundle. The point is that the open subsets $C^k\Gamma(E)$ of $C^k\Gamma(\eta)$, with their own structure of Banach space, give, through their embedding, the inverse of charts for that structure.

Let

$$\tau_\eta|_{TF\eta} : TF\eta \longrightarrow \eta, \quad TF_e \eta = T_e \eta_{\pi(e)}, \quad e \in \eta, \quad (63)$$

be the vector bundle of tangents to the fibers and let, for $\sigma \in C^k\Gamma(\eta)$,

$$\sigma^*(TF\eta) \longrightarrow D, \quad [\sigma^*(TF\eta)]_\zeta = T_{\sigma_\zeta} \eta_\zeta, \quad (64)$$

be the vector bundle pulled back through σ . As a consequence of the specific differential structure of $C^k\Gamma(\eta)$

$$\Phi_\sigma^k : T_\sigma C^k\Gamma(\eta) \xrightarrow{\sim} C^k\Gamma(\sigma^*(TF\eta)), \quad [\Phi_{\sigma(0)}^k(\frac{d\sigma(t)}{dt}|_{t=0})]_\zeta = \frac{d[\sigma(t)_\zeta]}{dt}|_{t=0}, \quad \zeta \in D, \quad (65)$$

is an isomorphism. On the other hand

$$\pi \circ \tau_\eta|_{TF\eta} : TF\eta \longrightarrow D, \quad (TF\eta)_\zeta = T(\eta_\zeta), \quad \zeta \in D, \quad (66)$$

is a nonlinear fiber bundle over D for which the manifold of sections $C^k\Gamma(TF\eta)$ becomes a vector bundle over $C^k\Gamma(\eta)$ through the projection

$$l(\tau_\eta|_{TF\eta}) : C^k\Gamma(TF\eta) \longrightarrow C^k\Gamma(\eta), \quad l(\tau_\eta|_{TF\eta})(\varphi) = \tau_\eta \circ \varphi, \quad (67)$$

$$C^k\Gamma(TF\eta)_\sigma = [l(\tau_\eta|_{TF\eta})]^{-1}(\{\sigma\}) = C^k\Gamma(\sigma^*(TF\eta)). \quad (68)$$

In this way Φ^k becomes an isomorphism of vector bundles, of Banach space fibers, over $C^k\Gamma(\eta)$:

$$\Phi^k : TC^k\Gamma(\eta) \xrightarrow{\sim} C^k\Gamma(TF\eta), \quad l(\tau_\eta|_{TF\eta}) \circ \Phi^k = \tau_{C^k\Gamma(\eta)}. \quad (69)$$

Remark that, if $E \subseteq \eta$ is a vector bundle neighbourhood in η , then TFE becomes a vector bundle neighbourhood for $TF\eta$ (again over D). Indeed,

$$TFE = (\tau_\eta|_{TF\eta})^{-1}(E) \quad (70)$$

is open in $TF\eta$ and its fiber $(TFE)_\zeta = T(E_\zeta)$ is the total space of the tangent bundle to the open subset E_ζ of η_ζ , a vector space itself. Recall the bijection $\Psi^E : E \times E \longrightarrow TE$, for an arbitrary vector space E (see (2)), that gives a structure of vector space on TE . Therefore $\Psi^{E_\zeta} : E_\zeta \times E_\zeta \longrightarrow T(E_\zeta)$, $\zeta \in D$, makes $E \times_D E$ a C^∞ vector bundle over D and

$$\Psi^E : E \times_D E \xrightarrow{\sim} TFE \quad (71)$$

a vector bundle isomorphism. In this way TFE is a vector bundle neighbourhood for $TF\eta$ if E is a vector bundle neighbourhood for η . And the open subset $C^k\Gamma(TFE)$ of

$C^k\Gamma(TF\eta)$ is also a Banach vector space with the same differentiable structure as that induced by the Banach manifold $C^k\Gamma(TF\eta)$. For the vector bundle E we have also the isomorphisms of Banach spaces

$$l(\Psi^E) : C^k\Gamma(E \times_D E) \xrightarrow{\sim} C^k\Gamma(TFE), \quad l(\Psi^E)(\sigma) = \Psi^E \circ \sigma, \quad (72)$$

$$\Sigma^k : C^k\Gamma(E) \times C^k\Gamma(E) \xrightarrow{\sim} C^k\Gamma(E \times_D E), \quad \Sigma^k(\sigma, \tau)_\zeta = (\sigma_\zeta, \tau_\zeta), \quad (73)$$

$$\Psi^{C^k\Gamma(E)} : C^k\Gamma(E) \times C^k\Gamma(E) \xrightarrow{\sim} TC^k\Gamma(E), \quad (74)$$

$$\Phi^k : TC^k\Gamma(E) \xrightarrow{\sim} C^k\Gamma(TFE), \quad (75)$$

that make a commutative diagram:

$$\Phi^k \cdot \Psi^{C^k\Gamma(E)} = l(\Psi^E) \cdot \Sigma^k. \quad (76)$$

Now, for every positive integer r we denote

$$C^\infty\Gamma_r(TC^k\Gamma(\eta)) = \{X \in C^\infty(C^{k+r}\Gamma(\eta), TC^k\Gamma(\eta)) \mid \tau_{C^k\Gamma(\eta)} \circ X = \text{id}_{C^{k+r}\Gamma(\eta)}\}. \quad (77)$$

We see that $C^\infty\Gamma_0(TC^k\Gamma(\eta)) = C^\infty\Gamma(TC^k\Gamma(\eta))$ in the usual sense of smooth vector fields on the manifold $C^k\Gamma(\eta)$. Next we consider

$$C^\infty\Gamma_r(TC^\infty\Gamma(\eta)) = \bigcap_{k=0}^{\infty} C^\infty\Gamma_r(TC^k\Gamma(\eta)). \quad (78)$$

Here we keep in mind that $C^\infty\Gamma(\eta)$ is dense in $C^k\Gamma(\eta)$, that Φ^k from (69) is independent of k in the sense that

$$\Phi^k|_{TC^{k+1}\Gamma(\eta)} = \Phi^{k+1}, \quad (79)$$

for the inclusion $TC^{k+1}\Gamma(\eta) \subseteq TC^k\Gamma(\eta)$ is tangent to the inclusion $C^{k+1}\Gamma(\eta) \subseteq C^k\Gamma(\eta)$. In this way we obtain

$$\Phi : \bigcap_{k=0}^{\infty} TC^k\Gamma(\eta) \longrightarrow C^\infty\Gamma(TF\eta) \quad (80)$$

and for $X \in C^\infty\Gamma_r(TC^\infty\Gamma(\eta))$ we get

$$\Phi \circ X : C^\infty\Gamma(\eta) \longrightarrow C^\infty\Gamma(TF\eta) \quad (81)$$

with the property that (see (69))

$$l(\tau_\eta|_{TF\eta}) \circ \Phi \circ X = \text{id}_{C^\infty\Gamma(\eta)}. \quad (82)$$

Finally we denote (see(78))

$$C^\infty\Gamma_\bullet(TC^\infty\Gamma(\eta)) = \bigcup_{r=0}^{\infty} C^\infty\Gamma_r(TC^\infty\Gamma(\eta)). \quad (83)$$

The elements of $C^\infty\Gamma_r(TC^\infty\Gamma(\eta))$ will be called smooth vector fields on $C^\infty\Gamma(\eta)$ of order r and those from $C^\infty\Gamma_\bullet(TC^\infty\Gamma(\eta))$ smooth vector fields of finite order. We show now that $C^\infty\Gamma_\bullet(TC^\infty\Gamma(\eta))$ becomes a graduated Lie algebra, in the sense that for a suitable bracket

$$[C^\infty\Gamma_r(TC^\infty\Gamma(\eta)), C^\infty\Gamma_s(TC^\infty\Gamma(\eta))] \subseteq C^\infty\Gamma_{r+s}(TC^\infty\Gamma(\eta)), \quad \forall r, s. \quad (84)$$

Let $X \in C^\infty\Gamma_r(TC^\infty\Gamma(\eta))$ and E be a vector bundle neighbourhood in η . As $C^k\Gamma(E)$ is open in $C^k\Gamma(\eta)$, for $\sigma \in C^k\Gamma(E)$, $T_\sigma C^k\Gamma(\eta) = T_\sigma C^k\Gamma(E)$ so that $X_\sigma \in T_\sigma C^k\Gamma(E)$ for $\sigma \in C^{k+r}\Gamma(E) \subseteq C^k\Gamma(E)$, we find that X maps $C^{k+r}\Gamma(E)$ into $TC^k\Gamma(E)$ for all k . If we denote

$$p_2^k : C^k\Gamma(E) \times C^k\Gamma(E) \longrightarrow C^k\Gamma(E) \quad (85)$$

the canonical projection, then the representation of X in the chart defined by E would be

$$\tilde{X}^k = p_2^k \circ (\Psi^{C^k\Gamma(E)})^{-1} \circ X. \quad (86)$$

Since $\Psi^{C^k\Gamma(E)}$, as well as p_2^k , are independent of k in the sense already defined (see(79)), it follows that $\tilde{X}^k|_{C^{k+r+1}\Gamma(E)} = \tilde{X}^{k+1}$, $\forall k \geq 0$. We get $\tilde{X} : C^\infty\Gamma(E) \longrightarrow C^\infty\Gamma(E)$ such that $\tilde{X} \in C^\infty(C^{k+r}\Gamma(E), C^k\Gamma(E))$, $\forall k \geq 0$. In fact, from (76) and (86) we get

$$\Phi(X_\sigma)_\zeta = \Psi^{E_\zeta}(\sigma_\zeta, \tilde{X}(\sigma)_\zeta). \quad (87)$$

Analogously, for $Y \in C^\infty\Gamma_s(TC^\infty\Gamma(\eta))$ we get $\tilde{Y} \in \bigcap_{k=0}^\infty C^\infty(C^{k+s}\Gamma(E), C^k\Gamma(E))$. Then for $\sigma \in C^{k+r+s}\Gamma(E)$

$$[\tilde{X}, \tilde{Y}](\sigma) = \langle \tilde{Y}'(\sigma); \tilde{X}(\sigma) \rangle - \langle \tilde{X}'(\sigma); \tilde{Y}(\sigma) \rangle \quad (88)$$

is well defined in $C^k\Gamma(E)$ and an easy inspection shows that in fact

$$[\tilde{X}, \tilde{Y}] \in \bigcap_{k=0}^\infty C^\infty(C^{k+r+s}\Gamma(E), C^k\Gamma(E)). \quad (89)$$

It remains to verify that, if we define

$$[X, Y]_\sigma = \Psi^{C^k\Gamma(E)}(\sigma, [\tilde{X}, \tilde{Y}](\sigma)) \quad (90)$$

for $\sigma \in C^{k+r+s}\Gamma(E)$, the definition does not depend on the vector bundle neighbourhood $E \supseteq \sigma(D)$. Let then E_1, E_2 be two such neighbourhoods of σ in η . We need to distinguish $\vartheta_j : E_j \longrightarrow \eta$, $j = 1, 2$, the respective fiber preserving embeddings and consider $\psi = \vartheta_2^{-1} \circ \vartheta_1$ the nonlinear fiber preserving mapping from an open subset $O_1 \subseteq E_1$ onto an open subset $O_2 \subseteq E_2$. Then $C^k\Gamma(O_j)$ are open in $C^k\Gamma(E_j)$ and $l(\psi) : C^k\Gamma(O_1) \longrightarrow C^k\Gamma(O_2)$, $l(\psi)(\sigma) = \psi \circ \sigma$, is a diffeomorphism for each $k \geq 0$. $l(\psi)$ is precisely the map that changes coordinates, in $C^k\Gamma(\eta)$, from $C^k\Gamma(E_1)$ into coordinates from $C^k\Gamma(E_2)$. But because $l(\psi)$ does not depend on k , all the computations go on as if all happens in the Banach spaces

$C^k\Gamma(E_j)$, $j = 1, 2$, for one fixed k . These considerations dealt with the graduated Lie algebra (83) of smooth vector fields on $C^\infty\Gamma(\eta)$ of finite order, for $\eta \rightarrow D$ denatural fiber bundle.

Now let us consider the case when $\eta = D \times M$ is the trivial nonlinear fiber bundle $p_D : D \times M \rightarrow D$. Then the identification of $\sigma \in C^k\Gamma(D \times M)$ with $\beta \in C^k(D, M)$ given by

$$C^k\Gamma(D \times M) \xrightarrow{\sim} C^k(D, M), \quad \sigma_\zeta = (\zeta, \beta(\zeta)), \quad \zeta \in D, \quad (91)$$

and that of

$$TF(D \times M) \xrightarrow{\sim} D \times TM, \quad T_{(\zeta, p)}(\{\zeta\} \times M) \xrightarrow{\sim} \{\zeta\} \times T_p M, \quad (92)$$

lead to the isomorphism of vector bundles over $C^k(D, M)$ (see (65), (69)):

$$\Phi^k : TC^k(D, M) \xrightarrow{\sim} C^k(D, TM), \quad (93)$$

$$\Phi_\beta^k : T_\beta C^k(D, M) \xrightarrow{\sim} C^k\Gamma(\beta^*(TM)), \quad (94)$$

$$[\Phi_{\beta(0)}^k(\frac{d\beta(t)}{dt}|_{t=0})]_\zeta = \frac{d[\beta(t)(\zeta)]}{dt}|_{t=0}, \quad \zeta \in D. \quad (95)$$

Still, in order to introduce charts on $C^k(D, M)$ in the neighbourhood of β_0 , we have to consider vector bundle neighbourhoods E of $\text{graph}(\beta_0)$ in $D \times M$. If χ is the chart defined by E , then as in (91)

$$\chi(\beta) = \sigma, \quad \sigma_\zeta = (\zeta, \beta(\zeta)), \quad \zeta \in D. \quad (96)$$

We will then denote

$$C^\infty\Gamma_\bullet(TC^\infty(D, M)) \quad (97)$$

the graduated Lie algebra of smooth vector fields on $C^\infty(D, M)$ of finite order.

There are two important instances of smooth vector fields of finite order on $C^\infty(D, M)$. First, for $X \in C^\infty\Gamma(TM)$

$$l(X)_\beta(\zeta) =: X_{\beta(\zeta)}, \quad \beta \in C^\infty(D, M), \quad \zeta \in D, \quad (98)$$

and second, for $X \in C^\infty\Gamma(TD)$

$$r(X)_\beta(\zeta) =: T_\zeta \beta \cdot X_\zeta, \quad \beta \in C^\infty(D, M), \quad \zeta \in D. \quad (99)$$

$l(X)$ is of order 0, while $r(X)$ is of order 1. And we have

$$l([X, Y]) = [l(X), l(Y)], \quad \forall X, Y \in C^\infty\Gamma(TM), \quad (100)$$

$$r([X, Y]) = -[r(X), r(Y)], \quad \forall X, Y \in C^\infty\Gamma(TD). \quad (101)$$

The first equality comes from the Lie group morphism

$$L : C^\infty\text{Diff} M \rightarrow C^\infty\text{Diff} C^k(D, M), \quad L(\varphi)(\beta) = \varphi \circ \beta, \\ \varphi \in C^\infty\text{Diff} M, \quad \beta \in C^k(D, M),$$

while from the Lie group antimorphism

$$R : C^\infty \text{Diff} D \longrightarrow C^\infty \text{Diff} C^k(D, M), \quad R(\varphi)(\beta) = \beta \circ \varphi, \\ \varphi \in C^\infty \text{Diff} D, \quad \beta \in C^k(D, M),$$

comes the second equality. Indeed, we have

$$l = T_{\text{id}_M} L, \quad r = T_{\text{id}_D} R$$

as the Lie algebra morphism, or antimorphism, corresponding to them. According to (93) and (67) we will understand

$$l(\tau_M) : C^k(D, TM) \longrightarrow C^k(D, M) \quad (102)$$

as a vector bundle, $\forall k \geq 0$. A more general case is the following: for any $\pi : G \longrightarrow M$, smooth vector bundle, $l(\pi) : C^k(D, G) \longrightarrow C^k(D, M)$ is the smooth vector bundle of Banach space fiber

$$C^k(D, G)_\beta = C^k \Gamma(\beta^*(G)). \quad (103)$$

When $\sigma : M \longrightarrow G$ is a section of π , $l(\sigma)$ is a section of $l(\pi)$, if $l(\sigma)(\beta) = \sigma \circ \beta$. We denote, as in (77)

$$C^\infty \Gamma_r(C^k(D, G)) = \{X \in C^\infty(C^{k+r}(D, M), C^k(D, G)) \mid l(\pi) \circ X = \text{id}_{C^{k+r}(D, M)}\}.$$

Then

$$C^\infty \Gamma_\bullet(C^\infty(D, G)) = \bigcup_{r=0}^{\infty} \bigcap_{k=0}^{\infty} C^\infty \Gamma_r(C^k(D, G)) \quad (104)$$

will be the space of smooth sections, of finite order, for the vector bundle

$$l(\pi) : C^\infty(D, G) \longrightarrow C^\infty(D, M). \quad (105)$$

We call (105) *the lift of the vector bundle* $\pi : G \longrightarrow M$ *to* $C^\infty(D, M)$. Of special interest is the case when $G = H \subseteq TM$ is the vector subbundle under study and also the case when $G = TM/H$. We remark that, if $H \subseteq TM$ is a vector subbundle, then $C^k(D, H)$ is a vector subbundle, over $C^k(D, M)$, of $C^k(D, TM)$, $\forall k \geq 0$. In the case that P is a smooth section of $T^*M \otimes TM$ such that P_m is a projection on H_m , $\forall m \in M$ (as, for instance, when P_m is the orthogonal projection on H_m defined by a smooth metric on M) we may consider for $X \in C^\infty \Gamma_\bullet(C^\infty(D, TM))$

$$l(P) \cdot X \in C^\infty \Gamma_\bullet(C^\infty(D, H)) \quad (106)$$

defined $\forall \beta \in C^\infty(D, M)$ and $\forall \zeta \in D$ by

$$(l(P) \cdot X)_\beta(\zeta) = l(P)_\beta(\zeta) \cdot X_\beta(\zeta) = P_{\beta(\zeta)} \cdot X_\beta(\zeta). \quad (107)$$

Using such a projection (106), we see that the isomorphism

$$TM \widetilde{\longrightarrow} H \times_M TM/H \quad (108)$$

entails the isomorphism

$$C^\infty(D, TM) \xrightarrow{\sim} C^\infty(D, H) \times_{C^\infty(D, M)} C^\infty(D, TM/H) \quad (109)$$

and then also that

$$Q_\beta := C^\infty(D, TM)_\beta / C^\infty(D, H)_\beta \xrightarrow{\sim} C^\infty(D, TM/H)_\beta, \quad (110)$$

$\forall \beta \in C^\infty(D, M)$. Let us recall the vector bundle morphism (21) $P^H : TM \longrightarrow TM/H$. Analogously, we can consider

$$P^{C^\infty(D, H)} : C^\infty(D, TM) \longrightarrow C^\infty(D, TM) / C^\infty(D, H). \quad (111)$$

If we define $l(P^H) : C^\infty(D, TM) \longrightarrow C^\infty(D, TM/H)$ by $l(P^H)(X) = P^H \circ X$ we see that

$$Q_\beta \cdot P_\beta^{C^\infty(D, H)} = l(P^H)_\beta. \quad (112)$$

Taking into account the equality (112) and the formula (20), the following result gives the curvature of the lift to $C^\infty(D, M)$ in terms of the curvature of the subbundle. It is also in agreement with the relation (100).

Theorem 2 . *Let $H \subseteq TM$ be a smooth vector subbundle, over M , of curvature tensor C^H and D be a smooth compact manifold. Then*

$$P_{\beta(\zeta)}^H[X, Y]_\beta(\zeta) = C_{\beta(\zeta)}^H(X_\beta(\zeta) \wedge Y_\beta(\zeta)), \quad (113)$$

$\forall X, Y \in C^\infty \Gamma_\bullet(C^\infty(D, H))$, $\forall \beta \in C^\infty(D, M)$, $\forall \zeta \in D$, the Lie bracket from the left being taken as for smooth vector fields of finite order on $C^\infty(D, M)$.

Proof. We intend to prove (113) for a fixed β_0 and a fixed ζ_0 . Let then E be a vector bundle neighbourhood of $\text{graph}(\beta_0)$ in $D \times M$ and $\iota : E \longrightarrow D \times M$ be the respective fiber preserving open embedding of it. If $\pi : E \longrightarrow D$ is the respective structural projection and $p_2 : D \times M \longrightarrow M$ is canonical, then for

$$\kappa := p_2 \circ \iota, \quad \kappa : E \longrightarrow M, \quad (114)$$

we have $\iota(e) = (\pi(e), \kappa(e))$, $\forall e \in E$. And for every $\zeta \in D$

$$\kappa_\zeta := \kappa|_{E_\zeta}, \quad \kappa_\zeta : E_\zeta \longrightarrow M, \quad (115)$$

is a diffeomorphism on an open subset of M . Let $k \geq 0$ be fixed and let

$$O^k = \{\beta \in C^k(D, M) \mid \text{graph}(\beta) \subset E\} \quad (116)$$

be the open domain of the chart on $C^k(D, M)$:

$$\chi : O^k \longrightarrow C^k \Gamma(E), \quad \chi(\beta)_\zeta = (\zeta, \beta(\zeta)), \quad \forall \zeta \in D, \quad (117)$$

(see also (96)). Therefore

$$\chi^{-1}(\sigma) = \kappa \circ \sigma, \quad \forall \sigma \in C^k\Gamma(E), \quad (118)$$

or, for a fixed ζ_0

$$\chi(\beta)_{\zeta_0} = \kappa_{\zeta_0}^{-1}(\beta(\zeta_0)), \quad \forall \beta \in O^k. \quad (119)$$

In what follows $\kappa_{\zeta_0}^{-1}$ will be considered as an E_{ζ_0} -valued chart on M and will be denoted

$$\mathring{\chi} := \kappa_{\zeta_0}^{-1}. \quad (120)$$

Then (119) reads

$$\chi(\beta)_{\zeta_0} = \mathring{\chi}(\beta(\zeta_0)), \quad \forall \beta \in O^k. \quad (121)$$

Let us consider the map

$$\delta_{\zeta_0} : C^k(D, M) \longrightarrow M, \quad \delta_{\zeta_0}(\beta) = \beta(\zeta_0). \quad (122)$$

It is easy to see that (see (94))

$$T_\beta \delta_{\zeta_0} \cdot X = X_{\zeta_0}, \quad \forall X \in C^k\Gamma(\beta^*(TM)). \quad (123)$$

Taking now the differential with respect to β in (121) we get

$$(d_\beta \chi \cdot X)_{\zeta_0} = d_{\beta(\zeta_0)} \mathring{\chi} \cdot X_{\zeta_0}, \quad \forall X \in C^k\Gamma(\beta^*(TM)), \quad (124)$$

since the mapping $: C^k\Gamma(E) \longrightarrow E_{\zeta_0}, \quad \sigma \mapsto \sigma_{\zeta_0}$, is linear and continuous. On the other hand, for a smooth vector field X of order r on $C^\infty(D, M)$, if $\widetilde{\chi_* X}$ is defined by

$$((\chi^{-1})^* X)_\sigma = (\sigma, \widetilde{\chi_* X}(\sigma)), \quad \forall \sigma \in C^{k+r}\Gamma(E), \quad (125)$$

then $\widetilde{\chi_* X} : C^{k+r}\Gamma(E) \longrightarrow C^k\Gamma(E)$ as a C^∞ map (see(86) and (87)). Also, as in (29)

$$\widetilde{\chi_* X}(\sigma) = d_{\chi^{-1}(\sigma)} \chi \cdot X_{\chi^{-1}(\sigma)}, \quad \forall \sigma \in C^{k+r}\Gamma(E). \quad (126)$$

Combining this with (124) and (121) we come to

$$\widetilde{\chi_* X}(\sigma)_{\zeta_0} = d_{\mathring{\chi}^{-1}(\sigma_{\zeta_0})} \mathring{\chi} \cdot (X_{\chi^{-1}(\sigma)})(\zeta_0), \quad \forall \sigma \in C^{k+r}\Gamma(E). \quad (127)$$

Next, for $X, Y \in C^\infty\Gamma_\bullet(TC^\infty(D, M))$ (see (97))

$$\widetilde{\chi_*[X, Y]} = [\widetilde{\chi_* X}, \widetilde{\chi_* Y}] \quad (128)$$

(compare with (32) as a consequence of the definition (88) for $[\tilde{X}, \tilde{Y}]$ and (90) for $[X, Y]$. Then from (127) we get

$$[\widetilde{\chi_* X}, \widetilde{\chi_* Y}](\sigma)_{\zeta_0} = d_{\mathring{\chi}^{-1}(\sigma_{\zeta_0})} \mathring{\chi} \cdot ([X, Y]_{\chi^{-1}(\sigma)})(\zeta_0), \quad \forall \sigma \in C^{k+r+s}\Gamma(E), \quad (129)$$

(if Y is of order s) and finally

$$\begin{aligned} d_{\beta_0(\zeta_0)} \overset{\circ}{\chi} \cdot ([X, Y]_{\beta_0})(\zeta_0) &= \\ &= \frac{d}{dt} [\widetilde{\chi_* Y}(\chi(\beta_0) + t \widetilde{\chi_* X}(\chi(\beta_0)))_{\zeta_0} - \widetilde{\chi_* X}(\chi(\beta_0) + t \widetilde{\chi_* Y}(\chi(\beta_0)))_{\zeta_0}]|_{t=0}. \end{aligned} \quad (130)$$

Let us consider the subspaces of E_{ζ_0} (see(4))

$$\widetilde{\overset{\circ}{\chi}_* H}_{\overset{\circ}{\chi}(m)} := d_m \overset{\circ}{\chi} \cdot H_m, \quad m \in \kappa_{\zeta_0}(E_{\zeta_0}). \quad (131)$$

In virtue of the hypothesis that $X, Y \in C^\infty \Gamma_\bullet(C^\infty(D, H))$ we have $(X_\beta)(\zeta_0), (Y_\beta)(\zeta_0) \in H_{\beta(\zeta_0)}, \forall \beta \in C^\infty(D, M)$. It results from (127) that

$$\widetilde{\chi_* X}(\sigma)_{\zeta_0}, \widetilde{\chi_* Y}(\sigma)_{\zeta_0} \in \widetilde{\overset{\circ}{\chi}_* H}_{\sigma_{\zeta_0}}, \quad \forall \sigma \in C^\infty \Gamma(E). \quad (132)$$

Then, for σ in a neighbourhood of $\chi(\beta_0)$ in $C^\infty \Gamma(E)$, σ_{ζ_0} remains in a given neighbourhood U of

$$\chi(\beta_0)_{\zeta_0} = \overset{\circ}{\chi}(\beta_0(\zeta_0)) \quad (133)$$

(see(121)). Let us denote

$$V = \widetilde{\overset{\circ}{\chi}_* H}_{\overset{\circ}{\chi}(\beta_0(\zeta_0))} \quad (134)$$

and choose $W \subset E_{\zeta_0}$ a supplementary for V subspace, i.e. such that $E_{\zeta_0} = V \oplus W$. Ignoring in notation a natural isomorphism, we will write simply

$$E_{\zeta_0} = V \times W. \quad (135)$$

Next, we choose the neighbourhood U of $\chi(\beta_0)_{\zeta_0}$ such that

$$\widetilde{\overset{\circ}{\chi}_* H}_{(x,y)} = \text{graph}(\Gamma(x, y)) \quad (136)$$

for $(x, y) \in U$ and certain $\Gamma(x, y) \in \text{Hom}(V, W)$ (compare with (6)). In this way $\overset{\circ}{\chi}$ becomes an adapted to H chart in the neighbourhood of $\beta_0(\zeta_0)$. In order to simplify notation we denote

$$(x_0, y_0) = \overset{\circ}{\chi}(\beta_0(\zeta_0)) \quad (137)$$

and then (see (134)) $\Gamma(x_0, y_0) = 0$. We will put also

$$\sigma_0 := \chi(\beta_0) \quad (138)$$

and then

$$(\sigma_0)_{\zeta_0} = (x_0, y_0). \quad (139)$$

We recall the notation $P_e^{\widetilde{\overset{\circ}{\chi}_* H}}$ from (35) and the relation (36). Then, if we apply $P_{(x_0, y_0)}^{\widetilde{\overset{\circ}{\chi}_* H}}$ on both sides of (130), we get

$$\begin{aligned} d_{\beta_0(\zeta_0)} \overset{\circ}{\chi} / H_{\beta_0(\zeta_0)} \cdot P_{\beta_0(\zeta_0)}^H([X, Y]_{\beta_0})(\zeta_0) &= \\ &= P_{(x_0, y_0)}^{\widetilde{\overset{\circ}{\chi}_* H}} \frac{d}{dt} [\widetilde{\chi_* Y}(\sigma_0 + t \widetilde{\chi_* X}(\sigma_0))_{\zeta_0} - \widetilde{\chi_* X}(\sigma_0 + t \widetilde{\chi_* Y}(\sigma_0))_{\zeta_0}]|_{t=0}. \end{aligned} \quad (140)$$

From (132) and (136) we infer the existence of smooth functions v_X, v_Y on a neighbourhood of σ_0 in $C^\infty\Gamma(E)$ taking values in V , such that

$$\widetilde{\chi_*X}(\sigma)_{\zeta_0} = (v_X(\sigma), \Gamma(\sigma_{\zeta_0}) \cdot v_X(\sigma)) \in V \times W, \quad (141)$$

and analogously for $\widetilde{\chi_*Y}(\sigma)_{\zeta_0}$. Then the right hand side of (140) is

$$\begin{aligned} & P_{(x_0, y_0)}^{\widetilde{\chi_*H}} [< \widetilde{\chi_*Y}'(\sigma_0)_{\zeta_0}; \widetilde{\chi_*X}(\sigma_0) > - < \widetilde{\chi_*X}'(\sigma_0)_{\zeta_0}; \widetilde{\chi_*Y}(\sigma_0) >] = \\ & = P_{(x_0, y_0)}^{\widetilde{\chi_*H}} ([< v_Y'(\sigma_0); \widetilde{\chi_*X}(\sigma_0) > - < v_X'(\sigma_0); \widetilde{\chi_*Y}(\sigma_0) >], \\ & \Gamma((\sigma_0)_{\zeta_0}) [< v_Y'(\sigma_0); \widetilde{\chi_*X}(\sigma_0) > - < v_X'(\sigma_0); \widetilde{\chi_*Y}(\sigma_0) >]) + \\ & + P_{(x_0, y_0)}^{\widetilde{\chi_*H}} (0, < \Gamma'(x_0, y_0); \widetilde{\chi_*X}(\sigma_0)_{\zeta_0} > v_Y(\sigma_0) - < \Gamma'(x_0, y_0); \widetilde{\chi_*Y}(\sigma_0)_{\zeta_0} > v_X(\sigma_0)) = \\ & = P_{(x_0, y_0)}^{\widetilde{\chi_*H}} (0, < \frac{\partial \Gamma}{\partial x}(x_0, y_0); v_X(\sigma_0) > v_Y(\sigma_0) - < \frac{\partial \Gamma}{\partial x}(x_0, y_0); v_Y(\sigma_0) > v_X(\sigma_0) + \\ & + < \frac{\partial \Gamma}{\partial y}(x_0, y_0); \Gamma(x_0, y_0)v_X(\sigma_0) > v_Y(\sigma_0) - < \frac{\partial \Gamma}{\partial y}(x_0, y_0); \Gamma(x_0, y_0)v_Y(\sigma_0) > v_X(\sigma_0)) = \\ & = \Lambda_{(x_0, y_0)}^W (< \frac{\partial \Gamma}{\partial x}(x_0, y_0); v_X(\sigma_0) > v_Y(\sigma_0) - < \frac{\partial \Gamma}{\partial x}(x_0, y_0); v_Y(\sigma_0) > v_X(\sigma_0) + \\ & + < \frac{\partial \Gamma}{\partial y}(x_0, y_0); \Gamma(x_0, y_0)v_X(\sigma_0) > v_Y(\sigma_0) - < \frac{\partial \Gamma}{\partial y}(x_0, y_0); \Gamma(x_0, y_0)v_Y(\sigma_0) > v_X(\sigma_0)) = \\ & = d_{\beta_0(\zeta_0)} \dot{\chi} / H_{\beta_0(\zeta_0)} \cdot C_{\beta_0(\zeta_0)}^H ((d_{\beta_0(\zeta_0)} \dot{\chi})^{-1} \Lambda_{\chi(\beta_0(\zeta_0))}^V \cdot v_X(\sigma_0) \wedge (d_{\beta_0(\zeta_0)} \dot{\chi})^{-1} \Lambda_{\chi(\beta_0(\zeta_0))}^V \cdot v_Y(\sigma_0)) = \\ & = d_{\beta_0(\zeta_0)} \dot{\chi} / H_{\beta_0(\zeta_0)} \cdot C_{\beta_0(\zeta_0)}^H (X_{\beta_0}(\zeta_0) \wedge Y_{\beta_0}(\zeta_0)) \end{aligned}$$

in virtue of (141), (127) and (40). Comparing with the left hand side of (140) we obtain (113) in β_0 and ζ_0 . The proof is complete ■

4 On a localization property

In the Theorem 4, §5, we state an equality of the form $[A, B] = C$ in the Lie algebra of vector fields of finite order over $C^\infty(D, M)$, for A, B, C certain vector fields of a simpler form. The equality is easy to prove for D and M diffeomorphic to open subsets of

vector spaces, but the argument for the possibility to localize the equality, even in this particular case, is quite involved. We preferred to allocate the entire section §4 to this question and to base our proof on the fact that A, B, C above belong to a Lie subalgebra of $C^\infty \Gamma_\bullet(TC^\infty(D, M))$, isomorphic to the set of global sections of a sheaf of Lie algebras over $D \times M$. On the other hand, the formula of the Lie bracket in this subalgebra, from Theorem 3 below, will be used further in section §6. But we presume that this Lie subalgebra and sheaf are interesting in themselves.

Coming back to the general case of a nonlinear fiber bundle $\pi : \eta \longrightarrow D$, $\zeta_0 \in D$ and $\sigma \in C^r \Gamma(\eta)$, $r \geq 0$, we denote $j_{\zeta_0}^r \sigma$ the r -jet of σ at ζ_0 and $J_{\zeta_0}^r \eta = \{j_{\zeta_0}^r \sigma \mid \sigma \text{ local section of } \eta \text{ around } \zeta_0\}$ (see Palais [5] for definitions). The disjoint union $J^r \eta$ becomes a fiber bundle over D of fiber $J_\zeta^r \eta$ at $\zeta \in D$ and $J^0 \eta = \eta$. On the other hand the projection

$$\pi_0^r : J^r \eta \longrightarrow \eta, \quad \pi_0^r(j_\zeta^r \sigma) = \sigma_\zeta, \quad (142)$$

makes $J^r \eta$ a fiber bundle over η of fiber

$$(J^r \eta)_e = \{j_\zeta^r \sigma \mid \zeta = \pi(e), \sigma_\zeta = e\}, \quad e \in \eta. \quad (143)$$

If

$$\pi^r : J^r \eta \longrightarrow D \quad (144)$$

denotes the canonical projection, then $\pi^0 = \pi$ and $\pi^r = \pi \circ \pi_0^r$. For $\sigma \in C^r \Gamma(\eta)$, $j^r \sigma$ becomes a section of $J^r \eta$ by $(j^r \sigma)_\zeta = j_\zeta^r \sigma$, $\zeta \in D$, such that $\forall k \geq 0$

$$j^r : C^{r+k} \Gamma(\eta) \longrightarrow C^k \Gamma(J^r \eta) \quad (145)$$

is a C^∞ map. Then if $\Xi : J^r \eta \longrightarrow TF\eta$ is a C^∞ fiber preserving map of fiber bundles over η , i.e.

$$\Xi_e : (J^r \eta)_e \longrightarrow T_e \eta_{\pi(e)}, \quad e \in \eta, \quad (146)$$

then

$$X_\sigma(\zeta) := \Xi_{\sigma_\zeta}(j_\zeta^r \sigma), \quad \zeta \in D, \sigma \in C^{r+k} \Gamma(\eta), \quad (147)$$

is a vector field of order r on $C^\infty \Gamma(\eta)$, i.e. $X \in C^\infty \Gamma_r(TC^\infty \Gamma(\eta))$. In this case we say that X is a vector field of differential order r . We denote $C^\infty FB_\eta(J^r \eta, TF\eta)$ the set of smooth maps Ξ of the form (146) (Since $\Xi : J^r \eta \longrightarrow TF\eta$ is also fiber preserving map over D , X defined by (147) is a nonlinear differential operator of order r from $C^\infty \Gamma(\eta)$ to $C^\infty \Gamma(TF\eta)$ in the sense of Palais [5]). Let next, for $0 \leq s \leq r$ (compare with (142))

$$\pi_s^r : J^r \eta \longrightarrow J^s \eta, \quad \pi_s^r(j_\zeta^r \sigma) = j_\zeta^s \sigma. \quad (148)$$

Being surjective, this fiber preserving map defines an embedding

$$C^\infty FB_\eta(J^s \eta, TF\eta) \hookrightarrow C^\infty FB_\eta(J^r \eta, TF\eta), \quad \Xi \mapsto \Xi \circ \pi_s^r, \quad (149)$$

that corresponds, through (147), to the embedding (see(78))

$$C^\infty \Gamma_s(TC^\infty \Gamma(\eta)) \subseteq C^\infty \Gamma_r(TC^\infty \Gamma(\eta)). \quad (150)$$

We may define accordingly the inductive limit

$$C^\infty FB_\eta(J^\bullet \eta, TF\eta) = \varinjlim_r C^\infty FB_\eta(J^r \eta, TF\eta) \quad (151)$$

and say that a vector field of the form (147) with $\Xi \in C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ is a vector field of *finite differential order* on $C^\infty \Gamma(\eta)$ and that Ξ is its *total symbol*. Our aim now is to show that these vector fields form a Lie subalgebra of the Lie algebra of vector fields of finite order, which means to identify a structure of graduated Lie algebra on $C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ for which the map $\Xi \mapsto X$ given by (147) becomes a morphism of graduated Lie algebras.

This structure enjoys of a certain localization property that makes the total symbols from $C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ global sections of a fine sheaf (see Warner [7]) of graduated Lie algebras over η . The use of this property needs a more tractable notion of fiber bundle and jets of sections. In fact, in what follows η will be a smooth manifold without boundary and $\pi : \eta \rightarrow D$ an arbitrary submersion. A local section will be a map $\sigma : U \rightarrow \eta$ defined on an open subset U of D such that $\pi \circ \sigma = \text{id}_U$. Then its jet $j_\zeta^k \sigma$ of order k is correctly defined for $\zeta \in U$. We define $TF_e \eta =: \ker T_e \pi$, $\forall e \in \eta$, and denote $\eta_\zeta =: \pi^{-1}(\{\zeta\})$, $\zeta \in D$. Then every open subset $\xi = \overset{\circ}{\xi} \subseteq \eta$ will be considered as a fiber bundle with the projection $\pi|_\xi$. In this case $(J^r \xi)_e = (J^r \eta)_e$, $TF_e \xi = TF_e \eta$, $\forall e \in \xi$ (see (143)). We may consider the vector spaces

$$V^r(\eta)_e := C^\infty((J^r \eta)_e, TF_e \eta), \quad r \geq 0, \quad (152)$$

$$V(\eta)_e := \varinjlim_r C^\infty((J^r \eta)_e, TF_e \eta), \quad e \in \eta, \quad (153)$$

and the associated vector bundle over η , so that every $\Xi \in C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ is a smooth section of the vector bundle $V(\eta)$. As, for ξ open in η and $e \in \xi$, we have $V^r(\xi)_e = V^r(\eta)_e$, $V(\xi)_e = V(\eta)_e$, the restriction $\Xi|_\xi \in C^\infty FB_\xi(J^\bullet \xi, TF\xi)$ has a clear meaning.

As it is shown in Palais [5], for every $F : \xi \rightarrow \eta$ fiber preserving map over D , there exists a well defined fiber preserving map (over D) $J^r(F) : J^r \xi \rightarrow J^r \eta$ by

$$J_\zeta^r(F)(j_\zeta^r \sigma) = j_\zeta^r(F \circ \sigma), \quad J_\zeta^r(F) = J^r(F)|_{J_\zeta^r \xi}. \quad (154)$$

In the case of the fiber preserving map over D (see(63)) $\tau_\eta|_{TF\eta} : TF\eta \rightarrow \eta$, the map

$$J_\zeta^r(\tau_\eta|_{TF\eta}) : J_\zeta^r(TF\eta) \rightarrow J_\zeta^r \eta \quad (155)$$

makes $J_\zeta^r(TF\eta)$ a fiber bundle over $J_\zeta^r \eta$ of fiber $(J_\zeta^r(TF\eta))_{j_\zeta^r \sigma} := [J_\zeta^r(\tau_\eta|_{TF\eta})]^{-1}(\{j_\zeta^r \sigma\})$. It is not difficult to verify that

$$(J_\zeta^r(TF\eta))_{j_\zeta^r \sigma} = J_\zeta^r(\sigma^*(TF\eta)). \quad (156)$$

Indeed, we remark first that in the definition of $j_\zeta^r \sigma$ enters only the values of σ on an arbitrary neighbourhood of ζ in D and similarly $J_\zeta^r \eta$ depends only on the restriction of η

to such a neighbourhood. On the other hand, the restriction of the fiber bundle $\sigma^*(TF\eta)$ to a neighbourhood of $\zeta \in D$ depends only on the values of σ on that neighbourhood. Then, in order to prove (156) we may first restrict to a neighbourhood of $\sigma(D)$ where η is a vector bundle and then to a neighbourhood of ζ where η is a trivial vector bundle. We will need in the sequel to identify the tangent space $T_{j_\zeta^r \sigma} J_\zeta^r \eta$ and in order of that we rephrase a result from Palais [5] (theorem 17.1, p.82) in a form more suitable for our purpose. Namely, we state it as an isomorphism of nonlinear fiber bundles over D

$$\Omega^r : TFJ^r\eta \xrightarrow{\sim} J^rTF\eta, \quad (157)$$

that is, given by diffeomorphisms depending smoothly on $\zeta \in D$

$$\Omega_\zeta^r : T(J_\zeta^r \eta) \xrightarrow{\sim} J_\zeta^r(TF\eta), \quad \zeta \in D, \quad (158)$$

and these, taking into account (156), as isomorphisms of vector bundles over $J_\zeta^r \eta$, being given by isomorphisms of vector spaces

$$(\Omega_\zeta^r)_{j_\zeta^r \sigma} : T_{j_\zeta^r \sigma} J_\zeta^r \eta \xrightarrow{\sim} J_\zeta^r(\sigma^*(TF\eta)) \quad (159)$$

depending smoothly of $j_\zeta^r \sigma \in J_\zeta^r \eta$ and $\zeta \in D$. More precisely

$$(\Omega_\zeta^r)_{j_\zeta^r \sigma} \left(\frac{d(j_\zeta^r \sigma(t))}{dt} \Big|_{t=0} \right) := j_\zeta^r \left(\frac{d\sigma(t)}{dt} \Big|_{t=0} \right), \quad (160)$$

where $\sigma(0) = \sigma$ so that in the right hand side $\frac{d\sigma(t)}{dt} \Big|_{t=0} \in C^r \Gamma(\sigma^*(TF\eta))$. To prove that $(\Omega_\zeta^r)_{j_\zeta^r \sigma}$, so defined by (160), is an isomorphism depending smoothly on $j_\zeta^r \sigma$ and ζ we may use again a local trivial and linear restriction of η . From the definition (160) we get (see (65))

$$(\Omega_\zeta^r)_{j_\zeta^r \sigma} \cdot T_\sigma j_\zeta^r = j_\zeta^r \cdot \Phi_\sigma^{k+r} \quad (161)$$

and

$$l(\Omega^r) \circ \Phi^k \circ Tj^r = j^r \circ \Phi^{k+r} \quad (162)$$

on $TC^{k+r}\Gamma(\eta)$. Here, in the left hand side Φ^k and j^r refer to η , while in the right hand side j^r and Φ^{k+r} refer to $TF\eta$.

In the formula of the Lie bracket in the space of total symbols $C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ we will use ω_M , the canonical involution on $T(TM)$, for an arbitrary manifold M (see, for instance, Abraham and Robbin [1]). We will also need the following explanatory

Proposition 6 . *Let M be a smooth manifold, $p \in M$, $u, v \in T_p M$ and $A, B \in T(TM)$ be such that*

$$A \in T_u TM, \quad B \in T_v TM, \quad \omega_M(A) \in T_v TM, \quad \omega_M(B) \in T_u TM. \quad (163)$$

Then the difference in $T_u TM$

$$\omega_M(B) - A \in T_u(T_p M), \quad (164)$$

as a subspace of $T_u TM$, and analogously the difference in $T_v TM$

$$\omega_M(A) - B \in T_v(T_p M). \quad (165)$$

Moreover, in $T_p M$ we have

$$[\Psi^{T_p M}(v, \cdot)]^{-1}(\omega_M(A) - B) = -[\Psi^{T_p M}(u, \cdot)]^{-1}(\omega_M(B) - A). \quad (166)$$

Proof. We know the important property (see [1])

$$T\tau_M \circ \omega_M = \tau_{TM}. \quad (167)$$

And also that

$$T_u(T_p M) = \ker T_u \tau_M. \quad (168)$$

From (163) we get $u = \tau_{TM}(A)$, $v = \tau_{TM}(B)$ and using also (167) we obtain $v = T\tau_M(A)$, $u = T\tau_M(B)$. Therefore $T_u \tau_M(\omega_M(B) - A) = \tau_{TM}(B) - T_u \tau_M(A) = v - v = 0$ and analogously $T_v \tau_M(\omega_M(A) - B) = \tau_{TM}(A) - T_v \tau_M(B) = u - u = 0$. To verify (166) we will use the covariance property

$$T(T\phi) \circ \omega_M = \omega_N \circ T(T\phi) \quad (169)$$

for any diffeomorphism $\phi : M \longrightarrow N$, the localization property

$$\omega_U = \omega_M|_{T(TU)} \quad (170)$$

for U open in M , and the formula for $M = V$ vector space

$$\omega_V(\Psi^{TV}(\Psi^V(x, y), \Psi^V(u, v))) = \Psi^{TV}(\Psi^V(x, u), \Psi^V(y, v)), \quad (171)$$

$\forall x, y, u, v \in V$. (Here TV with the structure of vector space induced by Ψ^V from $V \times V$). Using a local chart in the neighbourhood of $p \in M$ we are left to prove (166) in a vector space V . Let then $u = \Psi^V(p, s)$, $v = \Psi^V(p, t)$, $A = \Psi^{TV}(\Psi^V(p, s), \Psi^V(t, x))$, so that $B = \Psi^{TV}(\Psi^V(p, t), \Psi^V(s, y))$ (in virtue of (163) and (171)). Then $\omega_V(A) - B = \Psi^{TV}(\Psi^V(p, t), \Psi^V(s, x) - \Psi^V(s, y))$, where $\Psi^V(s, x) - \Psi^V(s, y) = \Psi^V(0, x - y)$, since the differences are taken in $T_v(TV)$, isomorphic to TV through $\Psi^{TV}(v, \cdot)$, and TV is isomorphic to $V \times V$ through Ψ^V . So that

$$\omega_V(A) - B = \Psi^{TV}(\Psi^V(p, t), \Psi^V(0, x - y)). \quad (172)$$

Remark now that

$$\Psi^{TV}(\Psi^V(p, t), \Psi^V(0, w)) = \Psi^{T_p V}(\Psi^V(p, t), \Psi^V(p, w)), \quad (173)$$

$\forall p, t, w \in V$. Indeed, $\Psi^{TV}(\Psi^V(p, t), \Psi^V(0, w)) = \frac{d}{dh}(\Psi^V(p, t) + h\Psi^V(0, w))|_{h=0} = \frac{d}{dh}\Psi^V(p, t + hw)|_{h=0} = \frac{d}{dh}(\Psi^V(p, t) + h\Psi^V(p, w))|_{h=0} = \Psi^{T_p V}(\Psi^V(p, t), \Psi^V(p, w))$, since the second time the sum is taken in $T_p V$, which is isomorphic through $\Psi^V(p, \cdot)$ to V . Combining (172) with (173) we get

$$[\Psi^{T_p V}(\Psi^V(p, t), \cdot)]^{-1}(\omega_V(A) - B) = \Psi^V(p, x - y). \quad (174)$$

Analogously, we get first

$$\omega_V(B) - A = \Psi^{TV}(\Psi^V(p, s), \Psi^V(0, y - x)) \quad (175)$$

and then

$$[\Psi^{T_p V}(\Psi^V(p, s), \cdot)]^{-1}(\omega_V(B) - A) = \Psi^V(p, y - x). \quad (176)$$

But in $T_p V$ we have $\Psi^V(p, y - x) = -\Psi^V(p, x - y)$ ■

Now for a nonlinear fiber bundle $\pi : \eta \longrightarrow D$, we consider the following

Axiom(A). *For every $\zeta_0 \in D$ and every smooth, locally defined around ζ_0 , section σ_0 of π there exists a smooth global section σ of π that coincides with σ_0 on an even smaller neighbourhood of ζ_0 .*

Remark that in both cases of a trivial nonlinear fiber bundle and that of a vector bundle the axiom (A) is satisfied. Of course, it is not satisfied by a fiber bundle without smooth global sections.

Let f_{sr} denote the canonical fiber preserving map (see Palais [5]) $f_{sr} : J^{s+r}\eta \longrightarrow J^s(J^r\eta)$

$$(f_{sr})_\zeta : J_\zeta^{s+r}\eta \longrightarrow J_\zeta^s(J^r\eta), \quad (f_{sr})_\zeta(j_\zeta^{s+r}\sigma) = j_\zeta^s(j^r\sigma); \quad (177)$$

it is distinct from $f_{rs} : J^{r+s}\eta \longrightarrow J^r(J^s\eta)$ defined by $(f_{rs})_\zeta(j_\zeta^{r+s}\sigma) = j_\zeta^r(j^s\sigma)$.

Theorem 3 . *Let us define, for $\Xi \in C^\infty FB_\eta(J^r\eta, TF\eta)$ and $\Theta \in C^\infty FB_\eta(J^s\eta, TF\eta)$,*

$$[\Xi, \Theta](j_\zeta^{r+s}\sigma) = [\Psi^{T_{\sigma_\zeta}\eta_\zeta}(\Xi(j_\zeta^r\sigma), \cdot)]^{-1} \cdot [\omega_{\eta_\zeta}((T\Theta \circ (\Omega^s)^{-1} \circ J^s(\Xi) \circ f_{sr})(j_\zeta^{r+s}\sigma)) - (T\Xi \circ (\Omega^r)^{-1} \circ J^r(\Theta) \circ f_{rs})(j_\zeta^{r+s}\sigma)], \quad (178)$$

*the difference being taken in $T_{\Xi(j_\zeta^r\sigma)}(T\eta_\zeta)$ with result in $T_{\Xi(j_\zeta^r\sigma)}(T_{\sigma_\zeta}\eta_\zeta)$ according to the previous **Proposition 6**. Then we have $\forall k, l \geq 0$ (see (148))*

$$[\Xi \circ \pi_r^{r+k}, \Theta \circ \pi_s^{s+l}] = [\Xi, \Theta] \circ \pi_{r+s}^{r+s+k+l} \quad (179)$$

which shows that the bracket (178) is well defined on $C^\infty FB_\eta(J^\bullet\eta, TF\eta)$ (see(151)) and that, for every ξ open in η ,

$$[\Xi, \Theta]|_\xi = [\Xi|_\xi, \Theta|_\xi]. \quad (180)$$

For any locally trivial fiber bundle η and for

$$X_\sigma(\zeta) := \Xi(j_\zeta^r \sigma), \quad Y_\sigma(\zeta) := \Theta(j_\zeta^s \sigma) \quad (181)$$

we have

$$[\Xi, \Theta](j_\zeta^{r+s} \sigma) = [X, Y]_\sigma(\zeta), \quad (182)$$

wherefrom we will infer that $[\Xi, \Theta]$ satisfies the Jacobi identity for all fiber bundle. Therefore $\Xi \mapsto X$ is a morphism of graduated Lie algebras from $C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ to $C^\infty \Gamma_\bullet(TC^\infty \Gamma(\eta))$, even a monomorphism in the case of an axiom **(A)** fiber bundle.

Proof. Let us verify first that (see(154));

$$(T\Theta \circ (\Omega^s)^{-1} \circ J^s(\Xi) \circ f_{sr})(j_\zeta^{r+s} \sigma) \in T_{\Theta(j_\zeta^s \sigma)}(T\eta_\zeta) \quad (183)$$

and that

$$\omega_{\eta_\zeta}((T\Theta \circ (\Omega^s)^{-1} \circ J^s(\Xi) \circ f_{sr})(j_\zeta^{r+s} \sigma)) \in T_{\Xi(j_\zeta^r \sigma)}(T\eta_\zeta). \quad (184)$$

For the first we see that $J^s(\Xi)(j_\zeta^r \sigma) = j_\zeta^s(\Xi(j_\zeta^r \sigma))$ and that $[(\Omega_\zeta^s)_{j_\zeta^s \sigma}]^{-1} j_\zeta^s(\Xi(j_\zeta^r \sigma)) \in T_{j_\zeta^s \sigma} J_\zeta^s \eta$. Finally $T\Theta(T_{j_\zeta^s \sigma} J_\zeta^s \eta) \subseteq T_{\Theta(j_\zeta^s \sigma)}(T\eta_\zeta)$, which proves (183). To prove (184) we apply on the left hand side $\tau_{T\eta_\zeta}$ aiming to obtain $\Xi(j_\zeta^r \sigma)$. We use then (167) and $\omega_M \circ \omega_M = \text{id}_{T(TM)}$, so that $\tau_{T\eta_\zeta} \circ \omega_{\eta_\zeta} = T\tau_{\eta_\zeta}$. We note the relation

$$\tau_{\eta_\zeta} \circ \Theta = \pi_0^s \quad (185)$$

(where $\pi_0^s = \pi_0^s(\eta) : J^s \eta \longrightarrow \eta$) and also that

$$T\pi_0^s(\eta) = \pi_0^s(TF\eta) \circ \Omega^s, \quad (186)$$

if $\pi_0^s(TF\eta) : J^s(TF\eta) \longrightarrow TF\eta$. In this way we obtain that $T\tau_{\eta_\zeta} \circ T\Theta \circ (\Omega^s)^{-1} = \pi_0^s(TF\eta)$, wherefrom the desired result. Analogously, simply by interchanging r with s and Ξ with Θ , we get

$$(T\Xi \circ (\Omega^r)^{-1} \circ J^r(\Theta) \circ f_{rs})(j_\zeta^{r+s} \sigma) \in T_{\Xi(j_\zeta^r \sigma)}(T\eta_\zeta) \quad (187)$$

and

$$\omega_{\eta_\zeta}((T\Xi \circ (\Omega^r)^{-1} \circ J^r(\Theta) \circ f_{rs})(j_\zeta^{r+s} \sigma)) \in T_{\Theta(j_\zeta^s \sigma)}(T\eta_\zeta), \quad (188)$$

which, together with Proposition 6, gives the meaning to the definition (178) (and shows moreover that $[\Xi, \Theta] = -[\Theta, \Xi]$ in this relation). Thus in order to verify (179) it is enough to show that

$$[\Xi \circ \pi_r^{r+k}, \Theta] = [\Xi, \Theta] \circ \pi_{r+s}^{r+s+k}. \quad (189)$$

But this is the easy consequence of the following commutation relations

$$f_{sr} \circ \pi_{s+r}^{s+r+k} = J^s(\pi_r^{r+k}) \circ f_{s,r+k}, \quad (190)$$

$$\Omega^r \circ T\pi_r^{r+k}(\eta) = \pi_r^{r+k}(TF\eta) \circ \Omega^{r+k} \quad (191)$$

(which extends (186)) and

$$\pi_r^{r+k}(TF\eta) \circ J^{r+k}(\Theta) \circ f_{r+k,s} = J^r(\Theta) \circ f_{rs} \circ \pi_{s+r}^{s+r+k}(\eta). \quad (192)$$

In proving (182) we have to consider a vector bundle neighbourhood E of σ . For the fiber bundles over η :

$$\pi_0^{r+s}(\eta) : J^{r+s}\eta \longrightarrow \eta, \quad (193)$$

$$\pi_0^r(\eta) \circ \pi_0^s(J^r\eta) : J^s(J^r\eta) \longrightarrow \eta, \quad (194)$$

$$\tau_\eta \circ \pi_0^s(TF\eta) : J^s(TF\eta) \longrightarrow \eta, \quad (195)$$

$$\pi_0^s(\eta) \circ \tau_{J^s\eta} : TF(J^s\eta) \longrightarrow \eta, \quad (196)$$

$$\tau_\eta \circ T\tau_\eta : TF(TF\eta) \longrightarrow \eta, \quad (197)$$

where

$$[TF(TF\eta)]_\zeta = T((TF\eta)_\zeta) = T(T(\eta_\zeta)), \quad (198)$$

the mappings

$$f_{sr} : J^{r+s}\eta \longrightarrow J^s(J^r\eta), \quad (199)$$

$$J^s(\Xi) : J^s(J^r\eta) \longrightarrow J^s(TF\eta), \quad (200)$$

$$(\Omega^s)^{-1} : J^s(TF\eta) \longrightarrow TF(J^s\eta), \quad (201)$$

$$T\Theta : TF(J^s\eta) \longrightarrow TF(TF\eta), \quad (202)$$

$$\omega_\eta : TF(TF\eta) \longrightarrow TF(TF\eta) \quad (203)$$

preserve fibers over η . We use here the fact that Ξ and Θ preserve fibers over η and that

$$\tau_\eta \circ T\tau_\eta = \tau_\eta \circ \tau_{T\eta}. \quad (204)$$

Analogously, we consider also Ω^r , $J^r(\Theta)$, $T\Xi$, f_{rs} as fiber preserving maps over η . If we use the generic notation $\pi_{F,\eta} : F(\eta) \longrightarrow \eta$ for any of the fiber bundles (193)-(197), we see that, for ξ open in η , $F(\xi) = \pi_{F,\eta}^{-1}(\xi)$, $\pi_{F,\xi} = \pi_{F,\eta}|_{F(\xi)}$. Taking into account that the mappings (199)-(203) preserve fibers over η , their restrictions to the fibers over ξ give the respective to ξ mappings. This proves (180). On the other hand, this shows that it is enough to prove (182) for $\eta = E$ vector bundle over D .

For X , Y given by (181) and \tilde{X} , \tilde{Y} defined according to (87) we consider $\tilde{\Xi}$, $\tilde{\Theta}$ such that

$$\tilde{\Xi}(j_\zeta^r\sigma) = \tilde{X}_\sigma(\zeta), \quad \tilde{\Theta}(j_\zeta^s\sigma) = \tilde{Y}_\sigma(\zeta). \quad (205)$$

Thus $\tilde{\Xi} : J^r E \longrightarrow E$ is defined by $\tilde{\Xi}$:

$$\tilde{\Xi}(j_\zeta^r\sigma) = \Psi^{E_\zeta}(\sigma_\zeta, \tilde{\Xi}(j_\zeta^r\sigma)), \quad (206)$$

being fiber preserving over D :

$$\tilde{\Xi}_\zeta =: \tilde{\Xi}|_{J_\zeta^r E}, \quad \tilde{\Xi}_\zeta : J_\zeta^r E \longrightarrow E_\zeta, \quad \zeta \in D. \quad (207)$$

Then $\tilde{X} : C^{k+r}\Gamma(E) \longrightarrow C^k\Gamma(E)$, $\tilde{Y} : C^{k+s}\Gamma(E) \longrightarrow C^k\Gamma(E)$, $\forall k \geq 0$ and

$$\begin{aligned} < \tilde{Y}'(\sigma); \tilde{X}(\sigma) >_{\zeta} = \frac{d}{dt} \tilde{Y}(\sigma + t\tilde{X}(\sigma))_{\zeta}|_{t=0} = \frac{d}{dt} \tilde{\Theta}_{\zeta}(j_{\zeta}^s \sigma + t j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)))|_{t=0} = \\ = < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) >. \end{aligned}$$

Here $\tilde{\Theta}_{\zeta} : J_{\zeta}^s E \longrightarrow E_{\zeta}$ acts between vector spaces and then $\tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma) \in \text{Hom}(J_{\zeta}^s E, E_{\zeta})$.

Therefore, in virtue of the definition (88), (90) we have $\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]$ and

$$[\tilde{X}, \tilde{Y}](\sigma)_{\zeta} = < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) > - < \tilde{\Xi}'_{\zeta}(j_{\zeta}^r \sigma); j_{\zeta}^r(\tilde{\Theta}(j^s \sigma)) >. \quad (208)$$

In the right hand side of (178) we have $\Xi(j_{\zeta}^r \sigma) \in T_{\sigma_{\zeta}} E_{\zeta}$, $\Xi(j_{\zeta}^r \sigma) =$

$$= \Psi^{E_{\zeta}}(\sigma_{\zeta}, \tilde{\Xi}(j_{\zeta}^r \sigma)) = \frac{d}{dt}(\sigma + t\tilde{\Xi}(j^r \sigma))_{\zeta}|_{t=0}. \text{ Therefore } [(\Omega_{\zeta}^s)_{j_{\zeta}^s}]^{-1}(j_{\zeta}^s(\Xi(j^r \sigma))) =$$

$$= \frac{d}{dt}(j_{\zeta}^s \sigma + t j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)))|_{t=0} \text{ and } T_{j_{\zeta}^s \sigma} \Theta \cdot [(\Omega_{\zeta}^s)_{j_{\zeta}^s}]^{-1}(j_{\zeta}^s(\Xi(j^r \sigma))) =$$

$$= \frac{d}{dt} \Theta(j_{\zeta}^s \sigma + t j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)))|_{t=0} = \frac{d}{dt} \Psi^{E_{\zeta}}(\sigma_{\zeta} + t\tilde{\Xi}(j_{\zeta}^r \sigma), \tilde{\Theta}(j_{\zeta}^s \sigma + t j_{\zeta}^s(\tilde{\Xi}(j^r \sigma))))|_{t=0} =$$

$$= \frac{d}{dt} (\Psi^{E_{\zeta}}(\sigma_{\zeta}, \tilde{\Theta}(j_{\zeta}^s \sigma)) + t \Psi^{E_{\zeta}}(\tilde{\Xi}(j_{\zeta}^r \sigma), < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) >))|_{t=0} =$$

$$= \Psi^{TE_{\zeta}}(\Psi^{E_{\zeta}}(\sigma_{\zeta}, \tilde{\Theta}(j_{\zeta}^s \sigma)), \Psi^{E_{\zeta}}(\tilde{\Xi}(j_{\zeta}^r \sigma), < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) >))$$

in virtue of the linearity of $\Psi^{E_{\zeta}} : E_{\zeta} \times E_{\zeta} \longrightarrow TE_{\zeta}$. Therefore, with the definitions (154) and (177) we obtain

$$\omega_{\eta_{\zeta}}((T\Theta \circ (\Omega^s)^{-1} \circ J^s(\Xi) \circ f_{sr})(j_{\zeta}^{r+s} \sigma)) - (T\Xi \circ (\Omega^r)^{-1} \circ J^r(\Theta) \circ f_{rs})(j_{\zeta}^{r+s} \sigma) =$$

$$= \Psi^{TE_{\zeta}}(\Psi^{E_{\zeta}}(\sigma_{\zeta}, \tilde{\Xi}(j_{\zeta}^r \sigma)),$$

$$, \Psi^{E_{\zeta}}(0, < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) > - < \tilde{\Xi}'_{\zeta}(j_{\zeta}^r \sigma); j_{\zeta}^r(\tilde{\Theta}(j^s \sigma)) >)) =$$

$$= \Psi^{T_{\sigma_{\zeta}} E_{\zeta}}(\Psi^{E_{\zeta}}(\sigma_{\zeta}, \tilde{\Xi}(j_{\zeta}^r \sigma)),$$

$$, \Psi^{E_{\zeta}}(\sigma_{\zeta}, < \tilde{\Theta}'_{\zeta}(j_{\zeta}^s \sigma); j_{\zeta}^s(\tilde{\Xi}(j^r \sigma)) > - < \tilde{\Xi}'_{\zeta}(j_{\zeta}^r \sigma); j_{\zeta}^r(\tilde{\Theta}(j^s \sigma)) >)) =$$

$$= \Psi^{T_{\sigma_{\zeta}} E_{\zeta}}(\Xi(j_{\zeta}^r \sigma), \Psi^{E_{\zeta}}(\sigma_{\zeta}, [\tilde{X}, \tilde{Y}](\sigma)_{\zeta})) = \Psi^{T_{\sigma_{\zeta}} E_{\zeta}}(\Xi(j_{\zeta}^r \sigma), [X, Y]_{\sigma}(\zeta))$$

according to (208). We have used also (171) and (173). Finally, in virtue of the axiom **(A)** the map $\Xi \longmapsto X$ is injective. The injectivity, in turn, ensures that the bracket (178)

satisfies the Jacobi identity. This ends the proof of the Theorem ■

Remark. The formula (178) of the Lie bracket in $C^\infty FB_\eta(J^\bullet \eta, TF\eta)$ may be not of great value as such; it only shows that the vector fields of finite differential order form a Lie subalgebra in $C^\infty \Gamma_\bullet(TC^\infty \Gamma(\eta))$ and that the Lie bracket of their total symbols enjoys of the localization property (180). It results that the set of germs of sections of the vector bundle $V(\eta)$ (see (153)) bears a structure of sheaf of graduated Lie algebras. (The stalks are Lie algebras and not the fibers of $V(\eta)$, like in the case of the vector bundle $T\eta$ over η). The vector bundle $V(\eta)$ depends only on the completely integrable vector subbundle $TF\eta$ of $T\eta$ since $J^r \eta$ is determined by the local foliation given by $TF\eta$. It does not depend on the projection π ■

5 An additive formula for the curvature tensors of two supplementary subbundles

The previous digression is used, in particular, to argue the next result which refers to the case $\eta = D \times M$ as a fiber bundle over D , when $C^k \Gamma(D \times M) \xrightarrow{\sim} C^k(D, M)$, through the correspondence $\sigma \mapsto \beta$, $\sigma_\zeta = (\zeta, \beta(\zeta))$, $\zeta \in D$ (see(91)). In this case we use the notations

$$J^r(D, M) := J^r \eta, \quad (\zeta, j_\zeta^r \beta) = j_\zeta^r \sigma, \quad J^r(D, M)_{(\zeta, p)} = \{(\zeta, j_\zeta^r \beta) \mid \beta(\zeta) = p\}, \quad (209)$$

$$C^\infty FB_M(J^r(D, M), TM) := C^\infty FB_\eta(J^r \eta, TF\eta) \quad (210)$$

since in this case $J^r(D, M)$ becomes a fiber bundle over M through $(\zeta, j_\zeta^r \beta) \mapsto \beta(\zeta)$ and $TF_{(\zeta, p)}(D \times M) \cong \{0_{T_\zeta D}\} \times T_p M$ makes $TF(D \times M)$ a vector bundle over M . Thus for $\Xi \in C^\infty FB_M(J^r(D, M), TM)$ we have $\Xi(\zeta, j_\zeta^r \beta) \in T_{\beta(\zeta)} M$, $\forall \zeta \in D$, $\forall \beta \in C^r(D, M)$. Also, in this case, we denote

$$C^\infty FB_M(J^\bullet(D, M), TM) := C^\infty FB_\eta(J^\bullet \eta, TF\eta) \quad (211)$$

(see (151)) and

$$V^r(D, M)_{(\zeta, p)} = C^\infty(J^r(D, M)_{(\zeta, p)}, T_p M), \quad (212)$$

$$V(D, M)_{(\zeta, p)} = \varinjlim_r V^r(D, M)_{(\zeta, p)}. \quad (213)$$

For $A \in C^\infty \Gamma(T^*M \otimes TM)$ we denote

$$(l(A))_\beta(\zeta) := A_{\beta(\zeta)} \in T_{\beta(\zeta)}^* \otimes T_{\beta(\zeta)} M \quad (214)$$

and if $X \in C^\infty \Gamma(TD)$ we put

$$(l(A) \cdot r(X))_\beta(\zeta) = A_{\beta(\zeta)} \cdot T_\zeta \beta \cdot X_\zeta, \quad \zeta \in D, \quad \beta \in C^\infty(D, M) \quad (215)$$

(compare with (106)). In this way, $l(A) \cdot r(X)$ is a vector field of differential order 1 on $C^\infty(D, M)$.

Theorem 4 . For $X, Y \in C^\infty\Gamma(TD)$ and $A \in C^\infty\Gamma(T^*M \otimes TM)$ the following equality holds

$$[l(A) \cdot r(X), r(Y)] = -l(A) \cdot r([X, Y]) \quad (216)$$

as for vector fields on $C^\infty(D, M)$.

Proof. Remark that for $A_p = I_{T_p M}$, $\forall p \in M$, (216) becomes (109). The localization property of the Lie bracket, expressed by the relations (182) and (180), allow to reduce proving the identity on a product $G \times U$, where G and U are domains of chart in D and M respectively, that is, to show that $[l(A) \cdot r(X), r(Y)]_\beta(\zeta) = -A_{\beta(\zeta)} \cdot T_\zeta \beta \cdot [X, Y]_\zeta$, for $\zeta \in G$ and $\beta \in C^\infty(G, U)$. Taking into account the intrinsic geometric meaning of the operations, we may consider only the case when $G = \mathring{G} \subseteq L$, $U = \mathring{U} \subseteq V$, for L and V vector spaces. Then $A : U \longrightarrow V^* \otimes V$, $X, Y : G \longrightarrow L$ and $(l(A) \cdot r(X))_\beta(\zeta) = A(\beta(\zeta)) \cdot \beta'(\zeta) \cdot X(\zeta)$. In this case

$$\begin{aligned} &< (l(A) \cdot r(X))'_\beta(\beta); r(Y)_\beta > (\zeta) = < A'(\beta(\zeta)); r(Y)_\beta(\zeta) > \cdot \beta'(\zeta) \cdot X(\zeta) + \\ &+ A(\beta(\zeta)) \cdot < (r(Y)_\beta)'_\zeta(\zeta); X(\zeta) > = < A'(\beta(\zeta)); \beta'(\zeta)Y(\zeta) > \cdot \beta'(\zeta) \cdot X(\zeta) + \\ &+ A(\beta(\zeta)) \cdot \beta''(\zeta)(X(\zeta), Y(\zeta)) + A(\beta(\zeta)) \cdot < \beta'(\zeta); Y'(\zeta)X(\zeta) > \end{aligned}$$

and

$$\begin{aligned} &< r(Y)'_\beta(\beta); (l(A) \cdot r(X))_\beta > (\zeta) = < ((l(A) \cdot r(X))'_\beta)'_\zeta(\zeta); Y(\zeta) > = \\ &= < A'(\beta(\zeta)); \beta'(\zeta)Y(\zeta) > \cdot \beta'(\zeta) \cdot X(\zeta) + A(\beta(\zeta)) \cdot \beta''(\zeta)(Y(\zeta), X(\zeta)) + \\ &+ A(\beta(\zeta)) \cdot < \beta'(\zeta); X'(\zeta)Y(\zeta) >. \end{aligned}$$

Therefore

$$\begin{aligned} &[l(A) \cdot r(X), r(Y)]_\beta(\zeta) = < r(Y)'_\beta(\beta); (l(A) \cdot r(X))_\beta > (\zeta) - \\ &- < (l(A) \cdot r(X))'_\beta(\beta); r(Y)_\beta > (\zeta) = \\ &= A(\beta(\zeta)) \cdot < \beta'(\zeta); X'(\zeta)Y(\zeta) - Y'(\zeta)X(\zeta) > = -(l(A) \cdot r([X, Y]))_\beta(\zeta) \quad \blacksquare \end{aligned}$$

Corollary 1 . In the same conditions on X, Y and A we have

$$\begin{aligned} [l(A) \cdot r(X), l(A) \cdot r(Y)] + l(A) \cdot r([X, Y]) &= [l(I - A) \cdot r(X), l(I - A) \cdot r(Y)] + \\ &+ l(I - A) \cdot r([X, Y]). \quad (217) \end{aligned}$$

Proof. From (216) we get also

$$[r(X), l(A) \cdot r(Y)] = -l(A) \cdot r([X, Y]). \quad (218)$$

As $l(I - A) \cdot r(X) = r(X) - l(A) \cdot r(X)$, (217) reduces to $l(A) \cdot r([X, Y]) = -[l(A) \cdot r(X), r(Y)] - [r(X), l(A) \cdot r(Y)] - l(A) \cdot r([X, Y])$ ■

The interior direct sum of two subspaces was denoted by

$$H \dot{+} K = V. \quad (219)$$

In this case

$$P_H^K : V \longrightarrow H \subseteq V \quad (220)$$

will stand for the canonical projection. Of course, in $V^* \otimes V$ we have

$$P_H^K + P_K^H = I_V. \quad (221)$$

Also

$$Q_H^K : V/K \xrightarrow{\sim} H \quad (222)$$

will denote the canonical isomorphism. In the case of two supplementary subbundles H and K of TM

$$H_m \dot{+} K_m = T_m M, \quad \forall m \in M, \quad (223)$$

P_H^K will denote the section of $T^*M \otimes TM$

$$(P_H^K)_m = P_{H_m}^{K_m}, \quad m \in M, \quad (224)$$

and Q_H^K the section of $\text{Hom}(TM/K, H)$

$$(Q_H^K)_m = Q_{H_m}^{K_m}, \quad m \in M. \quad (225)$$

Of course (see (21))

$$(Q_H^K)_m \cdot P_m^K = (P_H^K)_m, \quad \forall m \in M. \quad (226)$$

From Theorem 2 and Corollary 1 we obtain the important

Theorem 5 . *If H and K are two supplementary vector subbundles of TM*

$$\begin{aligned} & Q_{K\beta(\zeta)}^{H\beta(\zeta)} C_{\beta(\zeta)}^H (P_{H\beta(\zeta)}^{K\beta(\zeta)} T_\zeta \beta X_\zeta \wedge P_{H\beta(\zeta)}^{K\beta(\zeta)} T_\zeta \beta Y_\zeta) + \\ & + Q_{H\beta(\zeta)}^{K\beta(\zeta)} C_{\beta(\zeta)}^K (P_{K\beta(\zeta)}^{H\beta(\zeta)} T_\zeta \beta X_\zeta \wedge P_{K\beta(\zeta)}^{H\beta(\zeta)} T_\zeta \beta Y_\zeta) = \\ & = [l(P_K^H) \cdot r(X), l(P_K^H) \cdot r(Y)]_\beta(\zeta) + (l(P_K^H) \cdot r([X, Y]))_\beta(\zeta), \end{aligned} \quad (227)$$

for all $X, Y \in C^\infty \Gamma(TD)$, $\beta \in C^\infty(D, M)$, $\zeta \in D$.

Proof. In the Corollary 1 above we take $A = P_H^K$ and apply on both sides $P_{H\beta(\zeta)}^{K\beta(\zeta)}$ thus obtaining

$l(P_H^K) \cdot [l(P_H^K) \cdot r(X), l(P_H^K) \cdot r(Y)] + l(P_H^K) \cdot r([X, Y]) = l(P_H^K) \cdot [l(P_H^K) \cdot r(X), l(P_H^K) \cdot r(Y)]$. In this way, using Theorem 2, (226), (221) and Corollary 1 we get

$$\begin{aligned} Q_{H\beta(\zeta)}^{K\beta(\zeta)} C_{\beta(\zeta)}^K (P_{K\beta(\zeta)}^{H\beta(\zeta)} T_\zeta \beta X_\zeta \wedge P_{K\beta(\zeta)}^{H\beta(\zeta)} T_\zeta \beta Y_\zeta) &= P_{H\beta(\zeta)}^{K\beta(\zeta)} ([l(P_H^K) \cdot r(X), l(P_H^K) \cdot r(Y)]_\beta(\zeta) + \\ &+ (l(P_H^K) \cdot r([X, Y]))_\beta(\zeta)) = P_{H\beta(\zeta)}^{K\beta(\zeta)} ([l(P_H^K) \cdot r(X), l(P_H^K) \cdot r(Y)]_\beta(\zeta) + \\ &+ (l(P_H^K) \cdot r([X, Y]))_\beta(\zeta)). \end{aligned}$$

Transposing here H and K , adding these relations and using again (221) we obtain (227). The symmetry in the pair (H, K) , plain in the left hand side of (227), comes in the right hand side from Corollary 1 ■

Remark. The formula (227) may be read as an expression for the Lie bracket $[l(P_H^K) \cdot r(X), l(P_H^K) \cdot r(Y)]$; like (216) for $[l(A) \cdot r(X), r(Y)]$, it shows that this vector field is of differential order at most 1, when the general formula (178), of no use here, would give a differential order at most 2 ■

6 The parallel transport in a supplementary vector subbundle along a tangent path to the vector subbundle under study

Here P stands for the projection

$$P = P_K^H \quad (228)$$

corresponding to a direct sum decomposition (223). Conversely, starting from a smooth section P of $T^*M \otimes TM$ with $P_m^2 = P_m$, $\forall m$, we get the smooth subbundles H and K and the decomposition. Remark that the continuity in m ensures that P_m has a constant rank, since the rank of both P_m and $I - P_m$ may only increase in a neighbourhood of m . It is easy to show that *for a smooth section P in projections of $T_m M$, $\forall m \in M$, its lift to $T(TM)$ as*

$$\omega_M \circ TP \circ \omega_M \quad (229)$$

is again a linear projection in $T_X TM$ and moreover

$$\omega_M \circ TP \circ \omega_M + \omega_M \circ T(I - P) \circ \omega_M = I_{T_X TM}, \quad (230)$$

$\forall X \in TM$. Here P is considered as a smooth map

$$P : TM \longrightarrow TM, \quad P(X) = P_{\tau_M(X)} \cdot X, \quad (231)$$

so that $TP : T(TM) \longrightarrow T(TM)$. In what follows H and K (see (228)) will denote the total space of the respective vector bundles, hence the respective submanifolds of TM .

Theorem 6 . *Let H, K, P be as in (228) and smooth on M . Then the restriction of $T\tau_M + \tau_{TM}$ from $T(TM)$*

$$T\tau_M + \tau_{TM} : TK \cap \omega_M(TH) \longrightarrow TM \quad (232)$$

is a bijection of inverse R_P that verifies more precisely

$$\tau_{TM} \circ R_P = P, \quad T\tau_M \circ R_P = I - P. \quad (233)$$

Then, for all P

$$\omega_M \circ R_P = R_{(I-P)}. \quad (234)$$

It results that $T_k K \cap \omega_M(TH)$ is a vector subspace in $T_k K$ such that

$$R_P(k + (\cdot)) : H_{\tau_M(k)} \xrightarrow{\sim} T_k K \cap \omega_M(TH) \quad (235)$$

is a linear isomorphism and section for

$$T_k \tau_M : N_k \longrightarrow H_{\tau_M(k)}, \quad (236)$$

where

$$N_k =: T_k K \cap (T_k \tau_M)^{-1}(H_{\tau_M(k)}). \quad (237)$$

Finally, N_k is invariant for $\omega_M \circ T(I - P) \circ \omega_M$, and

$$\omega_M \circ T(I - P) \circ \omega_M|_{N_k} = R_P(k + (\cdot)) \cdot T_k \tau_M|_{N_k}, \quad (238)$$

$\forall k \in K$, *that is $\omega_M \circ T(I - P) \circ \omega_M$ projects N_k on $T_k K \cap \omega_M(TH)$. It results that $\omega_M \circ TP \circ \omega_M$ projects N_k on $T_k K_{\tau_M(k)}$.*

Proof. We start with $W, Z \in T(TM)$ such that

$$TP \cdot W = \omega_M(T(I - P) \cdot Z) \quad (239)$$

and show that, if $Y \in T(TM)$ satisfies

$$\omega_M(Y) = Y, \quad (240)$$

$$P(\tau_{TM}(Y)) = P(\tau_{TM}(W)), \quad (I - P)(\tau_{TM}(Y)) = (I - P)(\tau_{TM}(Z)) \quad (241)$$

then (see (239))

$$\begin{aligned} TP \cdot W &= \omega_M(T(I - P) \cdot Z) = \\ &= (TP \circ \omega_M \circ T(I - P))(Y) = (\omega_M \circ T(I - P) \circ \omega_M \circ TP)(Y). \end{aligned} \quad (242)$$

Remark that the condition (240) is equivalent to

$$\tau_{TM}(Y) = T\tau_M(Y) \quad (243)$$

and that for each $X \in TM$ there are (a diffeomorphic to the model vector space of M set of) Y -s such that

$$\tau_{TM}(Y) = T\tau_M(Y) = X. \quad (244)$$

Therefore there are as many Y -s satisfying (240) and (241). As the hypotheses (239)-(241) are local in character and intrinsic we may consider only the case when $M = U$ is open in V vector space and $P(x) \in V^* \otimes V$, $P(x)^2 = P(x)$, $\forall x \in U$. Let then

$$W = (x, y; u, v), \quad Z = (\xi, \eta; \varphi, \psi) \quad (245)$$

so that

$$TP \cdot W = (x, P(x)y; u, < P'(x); u > y + P(x)v), \quad (246)$$

$T(I - P) \cdot Z = (\xi, (I - P(\xi))\eta; \varphi, - < P'(\xi); \varphi > \eta + (I - P(\xi))\psi)$ and therefore

$$\omega_M(T(I - P) \cdot Z) = (\xi, \varphi; (I - P(\xi))\eta, - < P'(\xi); \varphi > \eta + (I - P(\xi))\psi). \quad (247)$$

Then the equality (239) gives

$$\xi = x, \quad \varphi = P(x)y, \quad u = (I - P(x))\eta, \quad (248)$$

$$< P'(x); (I - P(x))\eta > y + P(x)v = - < P'(x); P(x)y > \eta + (I - P(x))\psi.$$

Applying here $P(x)$ on both sides we get

$$P(x)v = -P(x) < P'(x); P(x)y > \eta - P(x) < P'(x); (I - P(x))\eta > y,$$

so that finally

$$TP \cdot W = (x, P(x)y; (I - P(x))\eta, (I - P(x)) < P'(x); (I - P(x))\eta > y - \\ - P(x) < P'(x); P(x)y > \eta). \quad (249)$$

From (248) we see also that

$$W = (x, y; (I - P(x))\eta, v), \quad Z = (x, \eta; P(x)y, \psi). \quad (250)$$

Therefore, with the definition (240)-(241) we find that

$$Y = (x, P(x)y + (I - P(x))\eta; P(x)y + (I - P(x))\eta, r) \quad (251)$$

with $r \in V$ arbitrary chosen. Then

$$T(I - P) \cdot Y = (x, (I - P(x))\eta; P(x)y + (I - P(x))\eta, \\ - < P'(x); P(x)y + (I - P(x))\eta > (P(x)y + (I - P(x))\eta) + (I - P(x))r), \\ \omega_M(T(I - P) \cdot Y) = (x, P(x)y + (I - P(x))\eta; (I - P(x))\eta,$$

$- < P'(x); P(x)y + (I - P(x))\eta > (P(x)y + (I - P(x))\eta) + (I - P(x))r$
and finally

$$\begin{aligned} TP(\omega_M(T(I - P) \cdot Y)) &= (x, P(x)y; (I - P(x))\eta, \\ &< P'(x); (I - P(x))\eta > (P(x)y + (I - P(x))\eta) - \\ &- P(x) < P'(x); P(x)y + (I - P(x))\eta > (P(x)y + (I - P(x))\eta). \end{aligned} \quad (252)$$

Let us compare now (252) with (249). Taking the derivative in the identity $P(x)^2 = P(x)$ we get first

$$P(x) < P'(x); a > P(x)b = 0, \quad \forall a, b \in V, \quad (253)$$

and using it for $I - P$ instead of P

$$(I - P(x)) < P'(x); a > (I - P(x))b = 0. \quad (254)$$

These are the keys to check the equality of the fourth terms in the expressions of $TP \cdot W$ and $TP(\omega_M(T(I - P) \cdot Y))$. If we transpose now W with Z and also P with $I - P$, we see that the hypothesis (239) is still satisfied and according to it definition (240)-(241) works with the same Y . Therefore Y will verify also $(T(I - P) \circ \omega_M \circ TP)(Y) = T(I - P) \cdot Z = \omega_M(TP \cdot W)$, wherefrom the third equality in (242).

We prove now that, as soon as $Y \in T(TM)$ satisfies (240) it also verifies

$$(TP \circ \omega_M \circ T(I - P))(Y) = (\omega_M \circ T(I - P) \circ \omega_M \circ TP)(Y). \quad (255)$$

Likewise, if locally,

$$Y = (x, y; y, z), \quad x \in U, \quad y, z \in V, \quad (256)$$

comparing with (252) we get

$$\begin{aligned} (TP \circ \omega_M \circ T(I - P))(Y) &= ((x, P(x)y; (I - P(x))y, < P'(x); (I - P(x))y > y + \\ &+ P(x) < (I - P)'(x); y > y \end{aligned} \quad (257)$$

and then interchanging P and $I - P$ and applying ω_M on both sides we come to

$$\begin{aligned} (\omega_M \circ T(I - P) \circ \omega_M \circ TP)(Y) &= (x, P(x)y; (I - P(x))y, < (I - P)'(x); P(x)y > y + \\ &+ (I - P(x)) < P'(x); y > y. \end{aligned} \quad (258)$$

It is easy to check the identity of the fourth terms in the right hand sides of these last two relations.

Let us define then, for $X \in TM$:

$$R_P(X) = (TP \circ \omega_M \circ T(I - P))(Y) = (\omega_M \circ T(I - P) \circ \omega_M \circ TP)(Y) \quad (259)$$

if $Y \in T(TM)$ satisfies

$$T\tau_M(Y) = \tau_{TM}(Y) = X. \quad (260)$$

From the computations above it results that $R_P(X)$ does not depend on Y chosen with (260) (see (256), (257)), that $R_P : TM \longrightarrow TK \cap \omega_M(TH)$, is surjective and satisfies (233) and (234). We see also that R_P is given locally by

$$R_P(x; y) = ((x, P(x)y; (I - P(x))y, < P'(x); (I - P(x))y > y + \\ + P(x) < (I - P)'(x); y > y \quad (261)$$

(see (257)). Since $\tau_{TM} + T\tau_M : T_k K \cap \omega_M(TH) \longrightarrow \{k\} + H_{\tau_M(k)}$ is a bijection and $T\tau_M$ is linear, it results that $T_k K \cap \omega_M(TH)$ is a vector subspace of $T_k K$ and that

$$T_k \tau_M : T_k K \cap \omega_M(TH) \xrightarrow{\sim} H_{\tau_M(k)} \quad (262)$$

is an isomorphism of inverse $R_P(k + (\cdot))$ (see (235)). Let us verify now (238). If locally $k = (x; y)$ with $P(x)y = y$, we find (see (237)):

$$N_{(x; y)} = \{(x, y; z, w) \mid P(x)z = 0, (I - P(x))w = < P'(x); z > y\} \quad (263)$$

and, for $P(x)y = y$, $P(x)z = 0$:

$$(\omega_M \circ TP \circ \omega_M)(x, y; z, w) = (x, y; 0, w + < P'(x); y > z - < P'(x); z > y), \quad (264)$$

$$R_P(x; y + z) = (x, y; z, < P'(x); z > y - < P'(x); y > z) \quad (265)$$

(see (261)). Note that $(I - P(x)) < P'(x); y > z = 0$ and $P(x) < P'(x); z > y = 0$, in virtue of (253) and (254), and therefore in (264)

$$(I - P(x))(w + < P'(x); y > z - < P'(x); z > y) = 0.$$

The proof is complete ■

Our aim is to show that there exists a canonical linear connection on the restriction of the vector bundle K to a path tangent to H , therefore defining a parallel transport in K along it.

Recall that an Ehresmann connection on a nonlinear fiber bundle is given by the *horizontal vector bundle*, in fact a smooth supplementary vector subbundle to the tangent at the nonlinear fiber in the tangent to the total space of the fiber bundle. The vector subbundle of the tangents to the nonlinear fibers is also called the vertical vector bundle. If

$\pi : \eta \longrightarrow D$ is the nonlinear fiber bundle, the connection is then defined by the linear projections

$$V_e : T_e \eta \longrightarrow T_e \eta, \text{ Im } V_e = \text{Ker } T_e \pi = T_e \eta_{\pi(e)} \quad (266)$$

on the vertical tangent spaces, smoothly depending on $e \in \eta$.

In the case of a vector bundle $\pi : E \longrightarrow D$, the mapping $T\pi : TE \longrightarrow TD$ delimits itself a *tangent vector bundle structure* defined as follows: let $s : E \times_D E \longrightarrow E$ denote the sum in the fibers of E and $Ts : T(E \times_D E) \longrightarrow TE$. Remark that $T_{(e, f)}(E \times_D E) = T_e E \times_{T_p D} T_f E$, if $\pi(e) = \pi(f) = p$. Then $T_{(e, f)} s \cdot (X, Y)$ is defined in the case that $T_e \pi \cdot X = T_f \pi \cdot Y \in T_p D$, and is precisely the sum between X and Y in the fiber of $T\pi$.

The connection on $\pi : E \longrightarrow D$ is linear in the case that the vertical projection V enjoys

of a second linearity property in the following sense. We remark that, in virtue of (266) $T_e\pi \cdot V_eZ = 0_{T_{\pi(e)}D}$, $\forall Z \in T_eE$, and if $T_e\pi \cdot Z = X$ then $V_eZ \in (T\pi)^{-1}(\{0_{T_{\tau_D(X)}D}\})$ since $\tau_D(X) = \pi(e)$. So, to be linear the connection, V should be also linear

$$V : (T\pi)^{-1}(\{X\}) \longrightarrow (T\pi)^{-1}(\{0_{T_{\tau_D(X)}D}\}) \quad (267)$$

$\forall X \in TD$, with respect to the tangent vector bundle structure. The linear connection defines the covariant derivative of sections σ of π with respect to vectors $X \in T_pD$ by

$$(\nabla_X\sigma)_p = [\Psi^{E_p}(\sigma_p, \cdot)]^{-1} \cdot V_{\sigma_p} \cdot T_p\sigma \cdot X. \quad (268)$$

In this way, for $X \in T_pD$ we have $(\nabla_X\sigma)_p \in E_p$. Of course, for a vector field X we get a new section $\nabla_X\sigma$ of π . It satisfies

$$\nabla_{fX}\sigma = f \cdot \nabla_X\sigma \quad (269)$$

and

$$\nabla_X f\sigma = f \cdot \nabla_X\sigma + (L_X f) \cdot \sigma, \quad (270)$$

for f scalar function, and is biadditive with respect to X and σ . Conversely, such a bilinear operator with (269) and (270) defines the vertical projection V from (268) and thus the linear connection.

In our special case, the covariant derivative arises as the linearization of a vector field of finite differential order on $C^\infty(D, M)$ in a critical point. From the local definition (88) we see that, if σ_0 is such that

$$X_{\sigma_0} = 0_{T_{\sigma_0}C^\infty\Gamma(\eta)}, \quad (271)$$

it is defined its *linearization at the critical point* σ_0

$$X'(\sigma_0) : C^\infty\Gamma(\sigma_0^*(TF\eta)) \longrightarrow C^\infty\Gamma(\sigma_0^*(TF\eta)) \quad (272)$$

by

$$X'(\sigma_0) \cdot W = [Y, X]_{\sigma_0}, \quad Y_{\sigma_0} = W; \quad (273)$$

(see (65) where we identify $T_\sigma C^\infty\Gamma(\eta)$ with $C^\infty\Gamma(\sigma^*(TF\eta))$). This means that the definition does not depend on the vector field Y of finite order chosen as an extension of W in a neighbourhood of σ_0 .

Theorem 7 . *Let H, K, P be as before smooth on M and D be a smooth compact curve (i.e. $\dim D = 1$). If $\beta_0 : D \longrightarrow M$ is a smooth path tangent to H , i.e.*

$$T_\zeta\beta_0 \cdot \xi \in H_{\beta_0(\zeta)}, \quad \forall \zeta \in D, \quad \forall \xi \in T_\zeta D, \quad (274)$$

*then on the vector bundle $p : \beta_0^*K \longrightarrow D$ there is a canonical linear connection given by the horizontal vector subbundle*

$$HT_{(\zeta, k)}\beta_0^*K =: (T_{(\zeta, k)}q)^{-1}(T_kK \cap \omega_M(TH)), \quad (275)$$

where $q : \beta_0^* K \longrightarrow K$ is canonical (see(235)). Let ξ be any smooth vector field on D . For the vector field

$$(l(P) \cdot r(\xi))_{\beta}(\zeta) = P_{\beta(\zeta)} \cdot T_{\zeta} \beta \cdot \xi_{\zeta}, \quad \beta \in C^{\infty}(D, M), \quad \zeta \in D, \quad (276)$$

(see (215)) β_0 is a critical point, $C^{\infty}\Gamma(\beta_0^* K) \subseteq C^{\infty}\Gamma(\beta_0^*(TM))$ is an invariant subspace for the linearization $(l(P) \cdot r(\xi))'(\beta_0)$ and the covariant derivative corresponding to the linear connection (275) is

$$\nabla_{\xi} \sigma = (l(P) \cdot r(\xi))'(\beta_0) \cdot \sigma, \quad \forall \sigma \in C^{\infty}\Gamma(\beta_0^* K). \quad (277)$$

Proof. We start by computing the linearization $(l(P) \cdot r(\xi))'(\beta_0)$ using the definition (273). As $l(P) \cdot r(\xi)$ is a vector field of differential order 1 we will apply Theorem 3, having in mind to consider the extension Y of W also of finite differential order. Suppose then that in (178), written for $[\Theta, \Xi]$, interchanging also r with s , we have $X_{\sigma_0} = 0$, i.e. $\Xi(j_{\zeta}^r \sigma_0) = 0$, $\forall \zeta \in D$ (see also (181)). Then we find that $(T\Theta \circ (\Omega^s)^{-1})(j_{\zeta}^s(\Xi(j^r \sigma_0))) = 0_{T_{Y_{\sigma_0}(\zeta)}(T\eta_{\zeta})}$, hence zero in the space $T_{Y_{\sigma_0}(\zeta)}(T\eta_{\zeta})$ where the difference is taken. Therefore, in the case that $X_{\sigma_0} = 0$, we have

$$[\Theta, \Xi](j_{\zeta}^{r+s} \sigma_0) = [\Psi^{T_{\sigma_0} \eta_{\zeta}}(\Theta(j_{\zeta}^s \sigma_0), \cdot)]^{-1} \cdot \omega_{\eta_{\zeta}}((T\Xi \circ (\Omega^r)^{-1} \circ J^r(\Theta) \circ f_{rs})(j_{\zeta}^{r+s} \sigma_0)). \quad (278)$$

Consider now the case when $\eta = D \times M$, $\dim D = 1$, $r = 1$ and let $s = 0$, that is

$$Y_{\beta}(\zeta) = \Theta(\zeta, \beta(\zeta)), \quad \zeta \in D, \quad \beta(\zeta) \in M. \quad (279)$$

Remember that

$$X_{\beta}(\zeta) = P_{\beta(\zeta)} \cdot T_{\zeta} \beta \cdot \xi_{\zeta}. \quad (280)$$

The vector field ξ on D defines the fiber preserving mappings

$$\begin{aligned} \varepsilon_{\xi}^0 : J^1(D \times M) &\longrightarrow D \times TM, \\ (\varepsilon_{\xi}^0)_{\zeta} : J_{\zeta}^1(D \times M) &\longrightarrow TM, \quad (\varepsilon_{\xi}^0)_{\zeta}(j_{\zeta}^1 \beta) = T_{\zeta} \beta \cdot \xi_{\zeta}, \end{aligned} \quad (281)$$

$$\begin{aligned} \varepsilon_{\xi}^1 : J^1(D \times TM) &\longrightarrow D \times T(TM), \\ (\varepsilon_{\xi}^1)_{\zeta} : J_{\zeta}^1(D \times TM) &\longrightarrow T(TM), \quad (\varepsilon_{\xi}^1)_{\zeta}(j_{\zeta}^1 X) = T_{\zeta} X \cdot \xi_{\zeta}. \end{aligned} \quad (282)$$

Therefore

$$X_{\beta}(\zeta) = \Xi(j_{\zeta}^1 \beta), \quad \Xi = P \circ \varepsilon_{\xi}^0. \quad (283)$$

As $TF(D \times M) = D \times TM$ we have

$$\Omega_{\zeta}^1 : T(J_{\zeta}^1(D \times M)) \longrightarrow J_{\zeta}^1(D \times TM) \quad (284)$$

(see (158)). Then the following relation holds

$$T(\varepsilon_\xi^0)_\zeta \circ (\Omega_\zeta^1)^{-1} = \omega_M \circ (\varepsilon_\xi^1)_\zeta, \quad \forall \zeta \in D. \quad (285)$$

This can be easily verified first when $M = U$ open in a vector space V , wherefrom in the general case. Then in our case the formula (278) reads

$$\begin{aligned} [\Theta, \Xi](j_\zeta^1 \beta_0) &= [\Psi^{T_{\beta_0(\zeta)} M}(\Theta(\zeta, \beta_0(\zeta)), \cdot)]^{-1} \cdot \omega_M((TP \circ T(\varepsilon_\xi^0)_\zeta \circ (\Omega_\zeta^1)^{-1})(j^1(\Theta))(\zeta, \beta_0(\zeta))) = \\ &= [\Psi^{T_{\beta_0(\zeta)} M}(\Theta(\zeta, \beta_0(\zeta)), \cdot)]^{-1} \cdot (\omega_M \circ TP \circ \omega_M)((\varepsilon_\xi^1)_\zeta(j^1(\Theta(\zeta, \beta_0(\zeta)))) = \\ &= [\Psi^{T_{\beta_0(\zeta)} M}(\sigma_\zeta, \cdot)]^{-1} \cdot (\omega_M \circ TP \circ \omega_M)(T_\zeta \sigma \cdot \xi_\zeta) \end{aligned}$$

where

$$\sigma_\zeta = \Theta(\zeta, \beta_0(\zeta)) = Y_{\beta_0}(\zeta) \in T_{\beta_0} M. \quad (286)$$

We thus found, for $X = l(P) \cdot r(\xi)$ and $X_{\beta_0} = 0$

$$(X'(\beta_0) \cdot \sigma)(\zeta) = [\Psi^{T_{\beta_0(\zeta)} M}(\sigma_\zeta, \cdot)]^{-1} \cdot (\omega_M \circ TP \circ \omega_M) \cdot T_\zeta \sigma \cdot \xi_\zeta \quad (287)$$

for $\sigma \in C^\infty \Gamma(\beta_0^*(TM)) = T_{\beta_0} C^\infty(D, M)$.

In the case that $X_{\beta_0}(\zeta) = 0$, $\forall \zeta$, and $Y_{\beta_0}(\zeta) \in K_{\beta_0(\zeta)}$, $\forall \zeta$, from Theorem 2 we infer that $[X, Y]_{\beta_0}(\zeta) \in K_{\beta_0(\zeta)}$, $\forall \zeta \in D$. Hence $C^\infty \Gamma(\beta_0^* K)$ is invariant for $X'(\beta_0)$.

In the formula above, however, σ is seen as a function $\sigma : D \rightarrow K$, hence when σ is thought as a section of $p : \beta_0^* K \rightarrow D$ we have to consider instead $q \circ \sigma$. Next we verify that $T_{(\zeta, k)} q : T_{(\zeta, k)} \beta_0^* K \rightarrow N_k$, $\forall (\zeta, k) \in \beta_0^* K$. But N_k is invariant for $\omega_M \circ TP \circ \omega_M$, which projects N_k on $T_k K_{\beta_0(\zeta)}$, according to Theorem 6. And we have to take into account also the isomorphism $(TF_{(\zeta, k)} q)^{-1} : T_k K_{\beta_0(\zeta)} \xrightarrow{\sim} TF_{(\zeta, k)} \beta_0^* K$, in order to come back in $\beta_0^* K$. Finally, for $\sigma \in C^\infty \Gamma(\beta_0^* K)$ we may replace in (287) $[\Psi^{T_{\beta_0(\zeta)} M}(\sigma_\zeta, \cdot)]^{-1}$ by $[\Psi^{K_{\beta_0(\zeta)}}(\sigma_\zeta, \cdot)]^{-1}$ to get

$$(X'(\beta_0) \cdot \sigma)(\zeta) = [\Psi^{(\beta_0^* K)_\zeta}(\sigma_\zeta, \cdot)]^{-1} \cdot (TF_{\sigma_\zeta} q)^{-1} \cdot (\omega_M \circ TP \circ \omega_M) \cdot T_{\sigma_\zeta} q \cdot T_\zeta \sigma \cdot \xi_\zeta, \quad (288)$$

and then define

$$V_{(\zeta, k)} = (TF_{(\zeta, k)} q)^{-1} \cdot (\omega_M \circ TP \circ \omega_M) \cdot T_{(\zeta, k)} q, \quad (\zeta, k) \in \beta_0^* K. \quad (289)$$

The formula (288), of the form (268), gives indeed a covariant derivative on $\beta_0^* K \rightarrow D$, for the vertical projection $V_{(\zeta, k)}$ above, if this V is linear with respect to the tangent vector bundle structure (see (267)). We note first that, for any vector bundle morphism $\Phi : E \rightarrow F$, the tangent mapping $T\Phi : TE \rightarrow TF$ is a vector bundle morphism with respect to the tangent vector bundle structures; next, the fact that

$\omega_M : T(TM) \rightarrow T(TM)$ is an isomorphism between the two vector bundle structures delimited by τ_{TM} and $T\tau_M$ on $T(TM)$. In (289) $q : \beta_0^* K \rightarrow TM$ is a vector bundle morphism and then Tq is a vector bundle morphism from Tp to $T\tau_M$. Next, TP is a vector bundle morphism from τ_{TM} to τ_{TM} and then $\omega_M \circ TP \circ \omega_M$ is a vector bundle morphism from $T\tau_M$ to $T\tau_M$. Therefore $\omega_M \circ TP \circ \omega_M \circ Tq$ is a vector bundle morphism from Tp to $T\tau_M$. More precisely, it gives a morphism from $(Tp)^{-1}(\{\xi\})$ to $(T\tau_M)^{-1}(0_{T_{\beta_0(\zeta)} M})$, if

$\xi \in T_\zeta D$. As $\iota : K \longrightarrow TM$ is a vector bundle embedding, $(T\tau_M|_K)^{-1}(0_{T_{\beta_0(\zeta)}M})$ is a subspace in $(T\tau_M)^{-1}(0_{T_{\beta_0(\zeta)}M})$ with the respective vector space structure and contains the image of that morphism. We remark that $(T\tau_M|_K)^{-1}(0_{T_{\beta_0(\zeta)}M}) = T(K_{\beta_0(\zeta)})$. (Note that, in general, the vector space structure of TV , given by Ψ^V , for V vector space (see p. 24) coincides with the tangent vector space structure if $V = E_\zeta$ and $\pi : E \longrightarrow D$ is a vector bundle so that $TV = (T\pi)^{-1}(\{0_{T_\zeta D}\})$). We thus found that $\omega_M \circ TP \circ \omega_M \circ Tq$ gives a vector space morphism from $(Tp)^{-1}(\{\xi\})$ to $T(K_{\beta_0(\zeta)})$, if $\xi \in T_\zeta D$. As $K_{\beta_0(\zeta)} = (\beta_0^* K)_\zeta$, $V_{(\zeta,k)}$ is a morphism from $(Tp)^{-1}(\{\xi\})$ to $(Tp)^{-1}(\{0_{T_\zeta D}\})$.

The covariant derivative

$$(\nabla_\xi \sigma)_\zeta = [\Psi^{(\beta_0^* K)_\zeta}(\sigma_\zeta, \cdot)]^{-1} \cdot (TF_{\sigma_\zeta} q)^{-1} \cdot (\omega_M \circ TP \circ \omega_M) \cdot T_{\sigma_\zeta} q \cdot T_\zeta \sigma \cdot \xi_\zeta \quad (290)$$

corresponds to the horizontal subbundle

$$HT_{(\zeta,k)}\beta_0^* K = \text{Ker } V_{(\zeta,k)} \quad (291)$$

(see (289)). It remains only to identify this space from (235) and (238) of Theorem 6, getting so (275) (recall also (230)) ■

Remark. In spite of its form (287), $X'(\beta_0) \cdot \sigma$ does not give a covariant derivative on $\beta_0^*(TM) \longrightarrow D$ with respect to ξ , since, for $\sigma \in C^\infty\Gamma(\beta_0^*(TM))$ and f smooth scalar function, we have

$$(X'(\beta_0) \cdot (f \cdot \sigma))(\zeta) = f(\zeta) \cdot (X'(\beta_0) \cdot \sigma)(\zeta) + (L_\xi f)(\zeta) \cdot P(\sigma_\zeta),$$

where P acts on TM as in (231). Compare with the required property (270). Of course $P(\sigma_\zeta) = \sigma_\zeta$, for $\sigma_\zeta \in K_{\beta_0(\zeta)}$ ■

In the case of an Ehresmann connection on the nonlinear fiber bundle $\pi : \eta \longrightarrow B$, we consider the transversal vector subbundle

$$K_e := \ker T_e \pi = TF_e \eta = T_e \eta_{\pi(e)}, \quad e \in \eta, \quad (292)$$

to the horizontal vector subbundle H of $T\eta$ (in $M = \eta$). If $\beta_0 : I \longrightarrow \eta$ is a horizontal path (on a compact interval I of \mathbf{R}), i.e. $\dot{\beta}_0(t) \in H_{\beta_0(t)}, \forall t \in I$, we consider the path in the base B

$$\gamma := \pi \circ \beta_0 \quad (293)$$

and the parallel transport along it

$$\gamma_s^t : G_{\beta_0(s)} \longrightarrow G_{\beta_0(t)} \quad (294)$$

from a neighbourhood $G_{\beta_0(s)}$ of $\beta_0(s)$ in $\eta_{\gamma(s)}$ to a neighbourhood $G_{\beta_0(t)}$ of $\beta_0(t)$ in $\eta_{\gamma(t)}$, for $s, t \in I$. On the other hand, according to the previous Theorem 7, we have a linear parallel transport

$$\tau_s^t \in \text{Hom}(K_{\beta_0(s)}, K_{\beta_0(t)}), \quad s, t \in I, \quad (295)$$

determined by the linear connection in $\beta_0^* K \longrightarrow I$. Then we have the following result

Theorem 8 . *In the hypothesis (292), with the notations (293), (294) and (295), we have*

$$\tau_s^t = T_{\beta_0(s)}\gamma_s^t, \quad (296)$$

*as $K_{\beta_0(t)} = T_{\beta_0(t)}\eta_{\gamma(t)}$, $\forall t \in I$. It results that, for $\eta = E$ vector bundle with a linear connection and ∇ the corresponding covariant derivative, if we denote $\nabla^{\beta_0^*K}$ the covariant derivative (277) on $C^\infty\Gamma(\beta_0^*K)$, the following equality holds*

$$\Psi^{E_{\gamma(t)}}(\beta_0(t), \nabla_{\dot{\gamma}(t)}\alpha_{\gamma(t)}) = \nabla_{\frac{\partial}{\partial t}}^{\beta_0^*K}\Psi^{E_{\gamma(t)}}(\beta_0(t), \alpha_{\gamma(t)}) \quad (297)$$

for $\alpha \in C^\infty\Gamma(E)$. Note that $\Psi^{E_{\gamma(t)}}(\beta_0(t), \alpha_{\gamma(t)})$, $\Psi^{E_{\gamma(t)}}(\beta_0(t), \nabla_{\dot{\gamma}(t)}\alpha_{\gamma(t)}) \in T_{\beta_0(t)}E_{\gamma(t)} = K_{\beta_0(t)}$.

Proof. The multiplicative properties

$$\gamma_s^t = \gamma_u^t \circ \gamma_s^u, \quad \tau_s^t = \tau_u^t \cdot \tau_s^u, \quad s, u, t \in I, \quad (298)$$

allow to reduce proving (296) for s and t as close to each other as we want. In this case, we can take $\eta = D \times U$, D open in V , U open in W , V, W vector spaces, $B = D$, $\pi : D \times U \longrightarrow D$ canonical. Also $H_{(x,y)} = \text{graph } \Gamma(x, y)$, $\Gamma(x, y) \in \text{Hom}(V, W)$, as in (6). Then, if

$$\beta_0(t) = (x(t), y(t)), \quad \gamma(t) = x(t), \quad (299)$$

$\gamma_s^t(y)$ is the solution of the problem in W

$$\frac{\partial}{\partial t}\gamma_s^t(y) = \Gamma(x(t), \gamma_s^t(y)) \cdot \dot{x}(t), \quad \gamma_s^s(y) = y, \quad y \in U. \quad (300)$$

The hypothesis on β_0 being horizontal gives

$$\gamma_s^t(y(s)) = y(t), \quad \forall t. \quad (301)$$

Taking the derivative with respect to y in (300), in $y = y(s)$, and taking into account (301), we find that $w(t) := \langle \frac{\partial \gamma_s^t}{\partial y}(y(s)); w_0 \rangle$ is the unique solution of the problem

$$\dot{w}(t) = \langle \frac{\partial \Gamma}{\partial y}(x(t), y(t)); w(t) \rangle \cdot \dot{x}(t), \quad w(s) = w_0. \quad (302)$$

In this case

$K = \{(x, y; 0_V, w) \mid x \in D, y \in U, w \in W\}$, $H = \{(x, y; v, \Gamma(x, y)v) \mid x \in D, y \in U, v \in V\}$ and we find that

$$T_{(x,y;0_V,w)}K \cap \omega_{D \times U}(TH) = \{(x, y, 0_V, w; v, \Gamma(x, y)v, 0_V, \langle \frac{\partial \Gamma}{\partial y}(x, y); w > v \rangle) \mid v \in V\}. \quad (303)$$

Since in trivialization

$$\beta_0^* K \simeq \{(t, w) \mid t \in I, w \in W\} \quad (304)$$

we have

$$q(t, w) = (x(t), y(t); 0_V, w) \quad (305)$$

and

$$Tq(t, w; \theta, \kappa) = (x(t), y(t); 0_V, w; \theta \dot{x}(t), \theta \dot{y}(t), 0_V, \kappa). \quad (306)$$

Therefore, according to (275)

$$HT_{(t,w)} \beta_0^* K = \{(t, w; \theta, \theta < \frac{\partial \Gamma}{\partial y}(x(t), y(t)); w > \cdot \dot{x}(t) \mid \theta \in \mathbf{R}\}. \quad (307)$$

Then $(t, w(t))$ is horizontal in $\beta_0^* K$ if and only if $(1, \dot{w}(t)) \in HT_{(t,w(t))} \beta_0^* K$, that is, when $\dot{w}(t) = < \frac{\partial \Gamma}{\partial y}(x(t), y(t)); w(t) > \cdot \dot{x}(t)$. Comparing with (302), we get (296).

Recall that for a linear connection we have

$$\nabla_{\dot{\gamma}(t)} \alpha_{\gamma(t)} = \frac{d}{dh} \gamma_{t+h}^t \alpha_{\gamma(t+h)}|_{h=0}. \quad (308)$$

On the other hand, for $A \in \text{Hom}(V, W)$

$$T_v A \cdot \Psi^V(v, u) = \Psi^W(Av, Au), \quad \forall v, u \in V. \quad (309)$$

In this way

$$\begin{aligned} \Psi^{E_{\gamma(t)}}(\beta_0(t), \nabla_{\dot{\gamma}(t)} \alpha_{\gamma(t)}) &= \Psi^{E_{\gamma(t)}}(\beta_0(t), \frac{d}{dh} \gamma_{t+h}^t \alpha_{\gamma(t+h)}|_{h=0}) = \\ &= \frac{d}{dh} \Psi^{E_{\gamma(t)}}(\beta_0(t), \gamma_{t+h}^t \alpha_{\gamma(t+h)})|_{h=0} = \frac{d}{dh} \Psi^{E_{\gamma(t)}}(\gamma_{t+h}^t \beta_0(t+h), \gamma_{t+h}^t \alpha_{\gamma(t+h)})|_{h=0} = \\ &= \frac{d}{dh} T_{\beta_0(t+h)} \gamma_{t+h}^t \cdot \Psi^{E_{\gamma(t+h)}}(\beta_0(t+h), \alpha_{\gamma(t+h)})|_{h=0} = \\ &= \frac{d}{dh} \tau_{t+h}^t \Psi^{E_{\gamma(t+h)}}(\beta_0(t+h), \alpha_{\gamma(t+h)})|_{h=0} = \nabla_{\frac{\partial}{\partial t}}^{\beta_0} \Psi^{E_{\gamma(t)}}(\beta_0(t), \alpha_{\gamma(t)}), \end{aligned}$$

the outside derivatives being taken in $T_{\beta_0(t)} E_{\gamma(t)} = K_{\beta_0(t)}$. This ends the proof ■

Remark. We can take in (297)

$$\beta_0(t) = 0_{E_{\gamma(t)}}, \quad (310)$$

for $\gamma : I \longrightarrow B$ arbitrary, since the connection is linear. Then the equality shows that $\nabla_{\dot{\gamma}(t)}$ is determined by $\nabla_{\frac{\partial}{\partial t}}^{0_{E_{\gamma(t)}}}$, the connection given in Theorem 7 being thus the right

generalization of a linear connection on a vector bundle.

On the other hand, the local parallel transport maps (294) are still defined and the equality (296) holds under the only hypothesis that $C^K = 0$ in a neighbourhood in M of the tangent to H path $\beta_0(I)$ ■

Let us consider again $\pi : \eta \longrightarrow D$ a nonlinear fiber bundle, $B \subseteq D$ submanifold, both B and D compact with boundary, $\eta|_B$ and

$$\rho : C^\infty\Gamma(\eta) \longrightarrow C^\infty\Gamma(\eta|_B), \quad \rho(\sigma) := \sigma|_B, \quad (311)$$

the natural restriction. It is easy to verify that for $\sigma \in C^\infty\Gamma(\eta)$

$$\sigma^*(TF\eta)|_B = (\sigma|_B)^*(TF(\eta|_B)) \quad (312)$$

and that, for $k \geq 0$

$$T_\sigma \rho : T_\sigma C^k\Gamma(\eta) \longrightarrow T_{\sigma|_B} C^k\Gamma(\eta|_B), \quad T_\sigma \rho \cdot X = X|_B \quad (313)$$

with the same meaning. We will need in the sequel the following

Proposition 7 . *Let $X^D \in C^\infty\Gamma_\bullet(TC^\infty\Gamma(\eta))$, $X^B \in C^\infty\Gamma_\bullet(TC^\infty\Gamma(\eta|_B))$ be smooth vector fields of finite order such that*

$$X_{\sigma|_B}^B = X_\sigma^D|_B, \quad \forall \sigma \in C^\infty\Gamma(\eta). \quad (314)$$

If σ_0 is a critical point for X^D then $\sigma_0|_B$ is a critical point for X^B and

$$\langle (X^B)'(\sigma_0|_B); Z|_B \rangle = \langle (X^D)'(\sigma_0); Z \rangle|_B, \quad \forall Z \in C^\infty\Gamma(\sigma_0^*(TF\eta)). \quad (315)$$

Proof. We can use a vector bundle neighbourhood of $\sigma_0(D)$ and the corresponding neighbourhood and chart on $C^\infty\Gamma(\eta)$ ■

Note that this fact is an infinite dimensional analogue of the following:

Let M, N be smooth manifolds, $\rho : M \longrightarrow N$ smooth map, X^M, X^N smooth vector fields, on M, N respectively, such that

$$X_{\rho(x)}^N = T_x \rho \cdot X_x^M, \quad \forall x \in M. \quad (316)$$

If x_0 is a critical point for X^M , then $\rho(x_0)$ is a critical point for X^N and

$$(X^N)'(\rho(x_0)) \cdot T_{x_0} \rho = T_{x_0} \rho \cdot (X^M)'(x_0) \quad (317)$$

on $T_{x_0} M$.

Remember that we denote the covariant derivative (277) given by Theorem 7

$$\nabla_{\xi}^{\beta_0^* K} \sigma = (l(P) \cdot r(\xi))'(\beta_0) \cdot \sigma, \quad \forall \sigma \in C^\infty \Gamma(\beta_0^* K). \quad (318)$$

Also, if $D = I$, I interval in \mathbf{R}

$$(\beta_0^* K)_s^t \in \text{Hom}(K_{\beta_0(s)}, K_{\beta_0(t)}), \quad s, t \in I, \quad (319)$$

will stand for the parallel transport determined by this linear connection.

On the infinitesimal variation of the tangent paths to H we have first

Theorem 9 . *Let H and K be two supplementary subbundles of TM , I compact interval of \mathbf{R} , $s_0 \in \mathbf{R}$, $\varepsilon > 0$ and*

$$\beta : (s_0 - \varepsilon, s_0 + \varepsilon) \times I \longrightarrow M \quad (320)$$

be a smooth map such that

$$\frac{\partial \beta}{\partial t}(s, t) \in H_{\beta(s, t)}, \quad \forall s \in (s_0 - \varepsilon, s_0 + \varepsilon), \quad \forall t \in I. \quad (321)$$

Then

$$\nabla_{\frac{\partial}{\partial t}}^{\beta(s_0, \cdot)^* K} P_{K_{\beta(s_0, t)}}^{H_{\beta(s_0, t)}} \frac{\partial \beta}{\partial s}(s_0, t) = Q_{K_{\beta(s_0, t)}}^{H_{\beta(s_0, t)}} C_{\beta(s_0, t)}^H (P_{H_{\beta(s_0, t)}}^{K_{\beta(s_0, t)}} \frac{\partial \beta}{\partial s}(s_0, t) \wedge \frac{\partial \beta}{\partial t}(s_0, t)), \quad (322)$$

or, equivalently

$$\begin{aligned} \frac{\partial}{\partial t} \{ (\beta(s_0, \cdot)^* K)_t^{t_0} \cdot P_{K_{\beta(s_0, t)}}^{H_{\beta(s_0, t)}} \frac{\partial \beta}{\partial s}(s_0, t) \} = \\ = (\beta(s_0, \cdot)^* K)_t^{t_0} Q_{K_{\beta(s_0, t)}}^{H_{\beta(s_0, t)}} C_{\beta(s_0, t)}^H (P_{H_{\beta(s_0, t)}}^{K_{\beta(s_0, t)}} \frac{\partial \beta}{\partial s}(s_0, t) \wedge \frac{\partial \beta}{\partial t}(s_0, t)). \end{aligned} \quad (323)$$

Proof. We take in Theorem 5 $X = \frac{\partial}{\partial s}$, $Y = \frac{\partial}{\partial t}$ on suitable domain Ω with smooth boundary such that, for an $\eta > 0$, $(s_0 - \eta, s_0 + \eta) \times I \subset \bar{\Omega} \subset (s_0 - \varepsilon, s_0 + \varepsilon) \times I$ and consider $D = \bar{\Omega}$. In virtue of (321)

$$(l(P_K^H) \cdot r(\frac{\partial}{\partial t}))_{\beta}(s, t) = 0, \quad \forall (s, t) \in D, \quad (324)$$

and therefore (227) gives, for $\zeta = (s, t)$

$$[l(P_K^H) \cdot r(\frac{\partial}{\partial s}), l(P_K^H) \cdot r(\frac{\partial}{\partial t})]_{\beta}(s, t) = Q_{K_{\beta(s, t)}}^{H_{\beta(s, t)}} C_{\beta(s, t)}^H (P_{H_{\beta(s, t)}}^{K_{\beta(s, t)}} \frac{\partial \beta}{\partial s}(s, t) \wedge \frac{\partial \beta}{\partial t}(s, t)). \quad (325)$$

Now we apply the preceding Proposition 7 for $\eta = D \times M$, $B = \{(s_0, t) | t \in I\} \subset D$

$$X_{\beta}^D(s, t) = P_{K_{\beta(s, t)}}^{H_{\beta(s, t)}} \frac{\partial \beta}{\partial t}(s, t), \quad X_{\gamma}^B(t) = P_{K_{\gamma(t)}}^{H_{\gamma(t)}} \dot{\gamma}(t). \quad (326)$$

Then (315) gives

$$< (l(P_K^H) \cdot r(\frac{\partial}{\partial t}))'(\beta); Z > (s_0, t) = < (l(P_K^H) \cdot r(\frac{\partial}{\partial t}))'(\beta(s_0, \cdot)); Z|_{s=s_0} > (t). \quad (327)$$

According to Theorem 7

$$< (l(P_K^H) \cdot r(\frac{\partial}{\partial t}))'(\beta(s_0, \cdot)); Z|_{s=s_0} > (t) = \nabla_{\frac{\partial}{\partial t}}^{\beta(s_0, \cdot)^* K} Z_{(s_0, t)}. \quad (328)$$

We take here $Z_{(s, t)} = P_{K_{\beta(s, t)}}^{H_{\beta(s, t)}} \frac{\partial \beta}{\partial s}(s, t)$ and then (322) comes from (325), (327) and the definition (273). Finally, (322) and (323) are equivalent in virtue of the equality

$$\frac{\partial}{\partial t}(\gamma_t^{t_0} \alpha_{\gamma(t)}) = \gamma_t^{t_0} \nabla_{\dot{\gamma}(t)} \alpha_{\gamma(t)} \quad (329)$$

which holds for every linear connection and section α (see also (308)) ■

Coming back to the notations (219) - (222) and (21), i.e.

$$P^H : V \longrightarrow V/H \quad (330)$$

for the canonical projection, we recall (see, for instance, Narasimhan [4], for charts on Grassmann manifolds) the natural bijection between the supplementary subspaces K of H and the linear sections $S \in \text{Hom}(V/H, V)$ of P^H given by

$$K \widetilde{\mapsto} S = Q_K^H; \quad (331)$$

and also the affine structure of the space of these sections, modelled on the vector space $\text{Hom}(V/H, H)$, where

$$\overrightarrow{K_1 K_2} := Q_{K_2}^H - Q_{K_1}^H. \quad (332)$$

Note the equality

$$Q_{K_2}^H - Q_{K_1}^H = P_H^{K_1} Q_{K_2}^H. \quad (333)$$

In the framework of (319) we will denote

$$[\beta_0^* K]_s^t := (Q_{K_{\beta_0(t)}}^{H_{\beta_0(t)}})^{-1} \cdot (\beta_0^* K)_s^t \cdot Q_{K_{\beta_0(s)}}^{H_{\beta_0(s)}} \quad (334)$$

the operators induced in quotients

$$[\beta_0^* K]_s^t \in \text{Hom}(T_{\beta_0(s)} M / H_{\beta_0(s)}, T_{\beta_0(t)} M / H_{\beta_0(t)}), \quad s, t \in I. \quad (335)$$

They keep the multiplicative properties

$$[\beta_0^* K]_u^t \cdot [\beta_0^* K]_s^u = [\beta_0^* K]_s^t, \quad [\beta_0^* K]_t^s = ([\beta_0^* K]_s^t)^{-1}, \quad s, u, t \in I. \quad (336)$$

The operators $[\beta_0^* K]_s^t$ represent the parallel transport of the fibers of $\beta_0^*(TM/H) \rightarrow I$ determined by the linear connection obtained through the vector bundle isomorphism $Q_{K_{\beta_0(t)}}^{H_{\beta_0(t)}}$, $t \in I$, from the connection on $\beta_0^* K \rightarrow I$.

The operators $[\beta_0^* K]_s^t$ still depend on K , except for the case when the curvature of H is zero, as we can see from

Theorem 10 . *Let K^1 , K^2 be two smooth supplementary subbundles to the same vector subbundle H of TM and $\beta_0 : I \rightarrow M$ smooth map such that*

$$\dot{\beta}_0(t) \in H_{\beta_0(t)}, \quad \forall t \in I. \quad (337)$$

Then for any $t_0 \in I$ fixed and $v \in T_{\beta_0(t_0)}M/H_{\beta_0(t_0)}$ arbitrary

$$\frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^2]_{t_0}^t \cdot v \} = [\beta_0^* K^1]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^1} Q_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}} [\beta_0^* K^2]_{t_0}^t v \wedge \dot{\beta}_0(t)). \quad (338)$$

Proof. Remark first that, if (338) holds for some fixed t_0 and t and all v , then (338) holds for any other t_1 instead of t_0 , the same t and all $v \in T_{\beta_0(t_1)}M/H_{\beta_0(t_1)}$. To see this, it is enough to replace v by $[\beta_0^* K^2]_{t_1}^{t_0} v$, to apply on both sides $[\beta_0^* K^1]_{t_0}^{t_1}$ and use the multiplicative property (336). In this way, we may suppose in (338) t and t_0 as close to each other as we want - even equal.

The second step consists in proving (338) for K^2 defined by a submersion $\pi : U \rightarrow B$, U neighbourhood of $\beta_0(t_0)$,

$$K_p^2 = \ker T_p \pi, \quad \forall p \in U, \quad (339)$$

(as in (292)). We consider $v \in T_{\beta_0(t_0)}M/H_{\beta_0(t_0)}$ given and let $\delta(s)$ be a short path, defined for s around s_0 , such that

$$\pi(\delta(s)) = \pi(\beta_0(t_0)), \quad \forall s, \quad \delta(s_0) = \beta_0(t_0), \quad \dot{\delta}(s_0) = Q_{K_{\beta_0(t_0)}^2}^{H_{\beta_0(t_0)}} v. \quad (340)$$

Next, let $\gamma := \pi \circ \beta_0$ and

$$\beta(s, t) = \gamma_{t_0}^t(\delta(s)), \quad (341)$$

where $\gamma_{t_0}^t$ is the parallel transport of the nonlinear fibers $\pi^{-1}(\{b\})$, $b \in B$, along γ , given by the horizontal vector bundle H_p , $p \in U$. Then $\beta(s, t) \in \pi^{-1}(\{\gamma(t)\})$, for all s and all t , as $\beta(s, t_0) \in \pi^{-1}(\{\gamma(t_0)\})$ for all s (see (340)). Moreover

$$\beta(s_0, t) = \beta_0(t), \quad \forall t \in I, \quad (342)$$

since $\beta(s_0, t) = \gamma_{t_0}^t(\delta(s_0)) = \gamma_{t_0}^t(\beta_0(t_0)) = \beta_0(t)$, by the hypothesis (337). Therefore (341) gives a variation of β_0 through tangent to H paths. And

$$\frac{\partial \beta}{\partial s}(s_0, t) = Q_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}} [\beta_0^* K^2]_{t_0}^t v, \quad (343)$$

in virtue of Theorem 8, (296), (340) and definition (334). We apply Theorem 9 to H , $K = K^1$ and this β ; mutiplying both sides of (323) by $(Q_{K^1_{\beta_0(t_0)}}^{H_{\beta_0(t_0)}})^{-1}$ and using the equality $P_{K^1}^H \cdot Q_{K^2}^H = Q_{K^1}^H$, we obtain (338) for any K^1 and K^2 of the form (339). Third, we show that if (338) holds for K^1 and K^2 , it is verified also for the transposed pair, K^2 and K^1 . Indeed

$$\begin{aligned}
& \frac{d}{dt} \{ [\beta_0^* K^2]_t^{t_0} \cdot [\beta_0^* K^1]_t^{t_0} \cdot v \} = \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \}^{-1} v = \\
& = -[\beta_0^* K^2]_t^{t_0} \cdot [\beta_0^* K^1]_t^{t_0} \cdot \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \} \cdot [\beta_0^* K^2]_t^{t_0} \cdot [\beta_0^* K^1]_t^{t_0} v = \\
& = -[\beta_0^* K^2]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K^1_{\beta_0(t)}} Q_{K^2_{\beta_0(t)}}^{H_{\beta_0(t)}} [\beta_0^* K^1]_t^{t_0} v \wedge \dot{\beta}_0(t)) = \\
& = [\beta_0^* K^2]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K^2_{\beta_0(t)}} Q_{K^1_{\beta_0(t)}}^{H_{\beta_0(t)}} [\beta_0^* K^1]_t^{t_0} v \wedge \dot{\beta}_0(t)),
\end{aligned}$$

since from (333) we get $P_H^{K^2} Q_{K^1}^H = -P_H^{K^1} Q_{K^2}^H$. Then the equality (338) holds also for K^1 of the same form (339) and K^2 arbitrary.

Finally, we show that, if (338) holds for the pairs (K^1, K^3) and (K^3, K^2) , then it is verified by the pair (K^1, K^2) . Indeed,

$$\begin{aligned}
& \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \cdot v \} = \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^3]_t^{t_0} \cdot [\beta_0^* K^3]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \cdot v \} = \\
& = \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^3]_t^{t_0} \} \cdot [\beta_0^* K^3]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \cdot v + \\
& + [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^3]_t^{t_0} \cdot \frac{d}{dt} \{ [\beta_0^* K^3]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \cdot v \} = \\
& = [\beta_0^* K^1]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K^1_{\beta_0(t)}} Q_{K^3_{\beta_0(t)}}^{H_{\beta_0(t)}} [\beta_0^* K^2]_t^{t_0} v \wedge \dot{\beta}_0(t)) + \\
& + [\beta_0^* K^1]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K^3_{\beta_0(t)}} Q_{K^2_{\beta_0(t)}}^{H_{\beta_0(t)}} [\beta_0^* K^2]_t^{t_0} v \wedge \dot{\beta}_0(t)) = \\
& = [\beta_0^* K^1]_t^{t_0} \cdot C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K^1_{\beta_0(t)}} Q_{K^2_{\beta_0(t)}}^{H_{\beta_0(t)}} [\beta_0^* K^2]_t^{t_0} v \wedge \dot{\beta}_0(t)),
\end{aligned}$$

as (333) gives $P_H^{K^1} Q_{K^3}^H + P_H^{K^3} Q_{K^2}^H = P_H^{K^1} Q_{K^2}^H$. The proof ends by combining these facts

■

The equation (338) allows to determine the parallel transport of the connection induced on $\beta_0^* K^2 \rightarrow I$, when the operators $(\beta_0^* K^1)_s^t$, corresponding to another supplementary vector subbundle K^1 , are known.

7 The equation of infinitesimal variation through tangent paths to the given vector subbundle

First, the following consequence of the previous relation (338).

Theorem 11 . *In the same hypothesis (337), the operator*

$$(DX)_t =: [\beta_0^* K]_{t_0}^t \frac{d}{dt} ([\beta_0^* K]_t^{t_0} P^{H_{\beta_0(t)}} X_t) - C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}} X_t \wedge \dot{\beta}_0(t)), \quad (344)$$

acting as

$$D : C^\infty \Gamma(\beta_0^*(TM)) \longrightarrow C^\infty \Gamma(\beta_0^*(TM/H)), \quad (345)$$

depends neither on t_0 , nor on the smooth, supplementary to H , vector subbundle K .

Proof. Let K^1, K^2 be two such subbundles of TM ; then the difference of the expressions (344) corresponding to them gives

$$\begin{aligned} & [\beta_0^* K^1]_{t_0}^t \frac{d}{dt} ([\beta_0^* K^1]_t^{t_0} P^{H_{\beta_0(t)}} X_t) - C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^1} X_t \wedge \dot{\beta}_0(t)) - [\beta_0^* K^2]_{t_0}^t \frac{d}{dt} ([\beta_0^* K^2]_t^{t_0} P^{H_{\beta_0(t)}} X_t) + \\ & + C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^2} X_t \wedge \dot{\beta}_0(t)) = [\beta_0^* K^1]_{t_0}^t \cdot \frac{d}{dt} \{ [\beta_0^* K^1]_t^{t_0} \cdot [\beta_0^* K^2]_t^{t_0} \} \cdot [\beta_0^* K^2]_t^{t_0} P^{H_{\beta_0(t)}} X_t + \\ & + C_{\beta_0(t)}^H ((P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^2} - P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^1}) X_t \wedge \dot{\beta}_0(t)) = C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}^1} Q_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}} P^{H_{\beta_0(t)}} X_t \wedge \dot{\beta}_0(t)) + \\ & + C_{\beta_0(t)}^H ((P_{K_{\beta_0(t)}^1}^{H_{\beta_0(t)}} - P_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}}) X_t \wedge \dot{\beta}_0(t)) = C_{\beta_0(t)}^H ((Q_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}} - Q_{K_{\beta_0(t)}^1}^{H_{\beta_0(t)}}) P^{H_{\beta_0(t)}} X_t \wedge \dot{\beta}_0(t)) + \\ & + C_{\beta_0(t)}^H ((P_{K_{\beta_0(t)}^1}^{H_{\beta_0(t)}} - P_{K_{\beta_0(t)}^2}^{H_{\beta_0(t)}}) X_t \wedge \dot{\beta}_0(t)), \end{aligned}$$

where we have used (338) and also (333). But this makes zero, as

$$P_K^H = Q_K^H \cdot P^H \quad (346)$$

(see also (226)) ■

Next we prove the reciprocal of Theorem 9.

Theorem 12 . *If $\beta_0 : I \longrightarrow M$ is tangent to H , then $X \in C^\infty \Gamma(\beta_0^*(TM))$ satisfies*

$$(DX)_t = 0, \quad \forall t \in I, \quad (347)$$

if and only if there exists a smooth variation of β_0 on I

$$\beta : (-\varepsilon, \varepsilon) \times I \longrightarrow M, \quad \beta(0, \cdot) = \beta_0, \quad (348)$$

in tangent to H paths, i.e.

$$\frac{\partial \beta}{\partial t}(s, t) \in H_{\beta(s, t)}, \forall t \in I, \forall s \in (-\varepsilon, \varepsilon), \quad (349)$$

such that

$$X_t = \frac{\partial \beta}{\partial s}(0, t), \forall t \in I. \quad (350)$$

Proof. it is easy to verify that (323) can be written as $D \frac{\partial \beta}{\partial s}(0, \cdot) = 0$ on I .
Conversely, X being given with (347), we have to construct β with (348), (349) and (350).
First, we consider the case when

$$\beta_0(I) \subset U, \chi : U \longrightarrow V \times W, \chi(U) = D \times G, \quad (351)$$

χ being a chart of the form (5) where H is of the form (6) (D open in V , G open in W).
Then, disregarding the chart χ , we consider K and H as in (303), i.e.

$$K = \{(x, y; 0_V, w) \mid x \in D, y \in G, w \in W\}, H = \{(x, y; v, \Gamma(x, y)v) \mid x \in D, y \in G, v \in V\}. \quad (352)$$

Let us denote

$$\beta_0(t) = (x_0(t), y_0(t)) \in V \times W, X(t) = (A(t), B(t)) \in V \times W; \quad (353)$$

then the equation (347) becomes

$$\begin{aligned} \frac{dB}{dt}(t) - \Gamma(x_0(t), y_0(t)) \frac{dA}{dt}(t) - < \frac{\partial \Gamma}{\partial x}(x_0(t), y_0(t)); A(t) > \frac{dx_0}{dt}(t) - \\ - < \frac{\partial \Gamma}{\partial y}(x_0(t), y_0(t)); B(t) > \frac{dy_0}{dt}(t) = 0. \end{aligned} \quad (354)$$

If, taking into account (348), we denote

$$\beta(s, t) = (x(s, t), y(s, t)), x(0, t) = x_0(t), y(0, t) = y_0(t), \quad (355)$$

the condition (349) reads

$$\frac{\partial y}{\partial t}(s, t) = \Gamma(x(s, t), y(s, t)) \frac{\partial x}{\partial t}(s, t). \quad (356)$$

It is clear that (354) is the consequence of (356) when (350) is satisfied, i.e.

$$\frac{\partial x}{\partial s}(0, t) = A(t), \frac{\partial y}{\partial s}(0, t) = B(t). \quad (357)$$

So, taking (354) as hypothesis, together with

$$\frac{dy_0}{dt}(t) = \Gamma(x_0(t), y_0(t)) \frac{dx_0}{dt}(t), \quad (358)$$

we look for $x(s, t), y(s, t)$ with (356), (355) and (357). $x(s, t)$ can be chosen arbitrary with (355) and (357). Next, for $t_0 \in I$ fixed and $y(s, t_0)$ arbitrary such that

$$y(0, t_0) = y_0(t_0), \quad \frac{\partial y}{\partial s}(0, t_0) = B(t_0), \quad (359)$$

we consider $y(s, t)$ defined by (356) and $y(s, t_0)$ so prescribed. Then $y(s, t)$ will satisfy $y(0, t) = y_0(t)$ and $\frac{\partial y}{\partial s}(0, t) = B(t)$, $\forall t \in I$, since both $\frac{\partial y}{\partial s}(0, t)$ and $B(t)$ satisfy as $Z(t)$ the differential equation

$$\begin{aligned} \frac{dZ}{dt}(t) - < \frac{\partial \Gamma}{\partial y}(x_0(t), y_0(t)); Z(t) > \frac{dx_0}{dt}(t) - \Gamma(x_0(t), y_0(t)) \frac{dA}{dt}(t) - \\ - < \frac{\partial \Gamma}{\partial x}(x_0(t), y_0(t)); A(t) > \frac{dx_0}{dt}(t) = 0 \end{aligned}$$

with the same initial condition $Z(t_0) = B(t_0)$.

It remains however to show that $\exists \varepsilon > 0$ such that the solution of the equation (356) be defined on all of I for $|s| < \varepsilon$. We have in mind to use the following fact: *for a smooth vector field X , on a manifold M , of local flow $\varphi^t(x)$, $x \in M$ and t in a neighbourhood $J(x)$ of 0 in \mathbf{R} , $\varphi^0(x) = x$, if $\varphi^t(x_0)$ is defined for t in a compact interval I containing 0, there exists a neighbourhood U of x_0 in M such that $\varphi^t(x)$ is defined on all of I for $x \in U$.* (For M compact, $\varphi^t(x)$ is defined $\forall t \in \mathbf{R}$ and $\forall x \in M$. If M is not compact we consider $\psi \in C_0^\infty(M, \mathbf{R})$, $\psi = 1$ in a neighbourhood of $\{\varphi^t(x_0) \mid t \in I\}$. Then the flow of ψX is global and coincides with the flow of X on I for x in a neighbourhood of x_0 .) We write the equation (356), with $y(s, t_0)$ given, in the form

$$\begin{aligned} \frac{dz}{dt}(t) = \Gamma(x(\sigma(t), \tau(t)), z(t)) \frac{\partial x}{\partial \tau}(\sigma(t), \tau(t)), \quad \frac{d\sigma}{dt}(t) = 0, \quad \frac{d\tau}{dt}(t) = 1; \\ z(0) = y(s, t_0), \quad \sigma(0) = s, \quad \tau(0) = t_0. \end{aligned}$$

Then, of course, $\sigma(t) = s$, $\tau(t) = t + t_0$, $z(t) = y(s, t + t_0)$, $\forall t \in I - t_0$, and the solution will be defined on all of $I - t_0$ for s in a neighbourhood of 0.

In the general case, in order to find $\varepsilon > 0$ such that β is defined as in (348), with (349) and (350), satisfied on I , we proceed as follows. For all t_0 we find $J \ni t_0$ interval, open as a subset of the compact interval I , such that \bar{J} has the properties (351) of the interval I . We then extract a finite and minimal subcovering for I with such intervals J (that can be of one of the forms: $J = [a, t_2)$, $J = (t_1, t_2)$, $J = (t_1, b]$, $J = [a, b]$, if $I = [a, b]$). When the covering is minimal, the maps $J \mapsto \inf J$ and $J \mapsto \sup J$ are injective. Let us enumerate the intervals of the covering such that

$$\sup J_k < \sup J_{k+1}, \quad 1 \leq k < n, \quad (360)$$

if $n > 1$ is their number ($n = 1$ corresponds to the previous situation). It is easy to see that

$$\inf J_1 = a, \quad \sup J_n = b. \quad (361)$$

Also, to check the inequality

$$\inf J_{k+1} < \sup J_k, \quad 1 \leq k < n, \quad (362)$$

in the contrary case the points from $[\sup J_k, \inf J_{k+1}]$ being not covered.

Then the solution $\beta(s, t)$ can be constructed recurrently: first on $\overline{J_1}$ for $|s| < \varepsilon_1$; if the solution is already defined on $\overline{\bigcup_{k=1}^q J_k} = [a, \sup J_q]$, for $|s| < \varepsilon_q$, it can be extended to $\overline{\bigcup_{k=1}^{q+1} J_k}$, since according to (362) $\inf J_{q+1} < \sup J_q$ and on the interval $[\inf J_{q+1}, \sup J_q]$ the solution is already constructed using the chart χ_q . With $x(s, t)$ suitably extended, $y(s, \frac{1}{2}(\inf J_{q+1} + \sup J_q))$ inherited and the chart χ_{q+1} , we get the extension of the solution to $\overline{\bigcup_{k=1}^{q+1} J_k} = [a, \sup J_{q+1}]$, for a certain $\varepsilon_{q+1} \leq \varepsilon_q$ and $|s| < \varepsilon_{q+1}$; and so on, up to $q+1 = n$. The theorem is proven ■

Of course, the equation (347) can be written in the form (see (322))

$$\nabla_{\frac{\partial}{\partial t}}^{\beta_0^* K} P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = Q_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} C_{\beta_0(t)}^H (P_{H_{\beta_0(t)}}^{K_{\beta_0(t)}} X_t \wedge \dot{\beta}_0(t)). \quad (363)$$

The equation (347) appears to be the root for the *Jacobi equation of infinitesimal variation of geodesics*. In that case we have a linear connection on the vector bundle TM over a smooth manifold M . In this analysis we will make use of

Proposition 8 . *For every smooth path*

$$Z : I \longrightarrow TM, \quad (364)$$

on an interval $I \subseteq \mathbf{R}$, if we denote

$$\gamma := \tau_M \circ Z, \quad (365)$$

for $X \in TM$, $V_X : T_X(TM) \longrightarrow T_X(T_{\tau_M(X)}M)$ the vertical projection defining the connection (see (266)) and T the respective torsion tensor, the following equality holds $\forall t \in I$

$$\Psi^{T_{\gamma(t)}M}(\dot{\gamma}(t), \cdot)^{-1} V_{\dot{\gamma}(t)} \omega_M(\dot{Z}(t)) - \Psi^{T_{\gamma(t)}M}(Z(t), \cdot)^{-1} V_{Z(t)} \dot{Z}(t) = T_{\gamma(t)}(Z(t), \dot{\gamma}(t)). \quad (366)$$

Proof. As this equality (366) is local and of intrinsic meaning, we may suppose $M = U$ open subset in the vector space V , so that $TM = U \times V$. In that case

$$H_{(x,y)} \subset T_{(x,y)}(U \times V) = \{(x, y)\} \times V \times V,$$

$$H_{(x,y)} = \{(x, y; v, \Gamma(x, y) v) | v \in V\} \quad (367)$$

where the linearity of the connection corresponds to the linearity of $\Gamma(x, y)$ in y , or to the existence of linear $\Gamma(x)$ on $V \otimes V$ such that

$$\Gamma(x, y) v = \Gamma(x)(y \otimes v), \quad x \in U, \quad y, v \in V. \quad (368)$$

Then

$$V_{(x,y)}(x, y; v, w) = (x, y; 0_V, w - \Gamma(x)(y \otimes v)). \quad (369)$$

In these local coordinates the torsion tensor becomes (see Kobayashi + Nomizu, vol. I [2])

$$T_x(v, w) = \Gamma(x)(v \otimes w - w \otimes v). \quad (370)$$

For (see (364) and (365))

$$Z(t) = (x(t), y(t)), \quad \gamma(t) = x(t), \quad (371)$$

we have

$$\begin{aligned} \dot{\gamma}(t) &= (x(t); x'(t)), \quad \dot{Z}(t) = (x(t), y(t); x'(t), y'(t)), \\ \omega_M(\dot{Z}(t)) &= (x(t), x'(t); y(t), y'(t)). \end{aligned} \quad (372)$$

Using (369) we get

$$\begin{aligned} \Psi^{T_{\gamma(t)}M}(\dot{\gamma}(t), \cdot)^{-1} V_{\dot{\gamma}(t)} \omega_M(\dot{Z}(t)) &= (x(t); y'(t) - \Gamma(x(t))(x'(t) \otimes y(t))), \\ \Psi^{T_{\gamma(t)}M}(Z(t), \cdot)^{-1} V_{Z(t)} \dot{Z}(t) &= (x(t); y'(t) - \Gamma(x(t))(y(t) \otimes x'(t))), \end{aligned}$$

wherefrom the result ■

It is easy to establish *the relation between the Riemann tensor R and the curvature tensor C of the horizontal subbundle H of $T(TM)$ defining the linear connection*, namely

$$R_{\tau_M(X)}(T_X \tau_M A, T_X \tau_M B) X = -\Psi^{T_{\tau_M(X)}M}(X, \cdot)^{-1} Q_{K_X}^{H_X} C_X(A \wedge B), \quad (373)$$

if $X \in TM$, $A, B \in H_X$ and

$$K_X = T_X(T_{\tau_M(X)}M). \quad (374)$$

Recall that $T_X \tau_M : H_X \xrightarrow{\sim} T_{\tau_M(X)}M$, $\Psi^{T_{\tau_M(X)}M}(X, \cdot) : T_{\tau_M(X)}M \xrightarrow{\sim} K_X$ and $Q_{K_X}^{H_X} : T_X(TM)/H_X \xrightarrow{\sim} K_X$ are isomorphisms.

And the link with the Jacobi equation (see Kobayashi + Nomizu vol. II [2]) is given in

Theorem 13 . *Let M be smooth manifold with a linear connection on TM given by ∇ . Let $\gamma_0 : I \longrightarrow M$ be a geodesic for ∇ . Then $Z \in C^\infty \Gamma(\gamma_0^*(TM)) = T_{\gamma_0} C^\infty(I, M)$ is a solution for the Jacobi equation*

$$\nabla_{\dot{\gamma}_0(t)}^2 Z(t) + \nabla_{\dot{\gamma}_0(t)}(T(Z(t), \dot{\gamma}_0(t))) + R(Z(t), \dot{\gamma}_0(t)) \dot{\gamma}_0(t) = 0 \quad (375)$$

if and only if

$$X_t := \omega_M(\dot{Z}(t)) \quad (376)$$

satisfies the equation (347) for H the horizontal subbundle of $T(TM)$ corresponding to ∇ and

$$\beta_0 = \dot{\gamma}_0. \quad (377)$$

Proof. From (297), for $P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = \Psi^{E_{\gamma(t)}}(\beta_0(t), \alpha_{\gamma(t)})$ and $E = TM$, we obtain

$$\Psi^{T_{\gamma_0(t)}M}(\beta_0(t), \cdot)^{-1} \nabla_{\frac{\partial}{\partial t}}^{\beta_0^* K} P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = \nabla_{\dot{\gamma}_0(t)} \Psi^{T_{\gamma_0(t)}M}(\beta_0(t), \cdot)^{-1} P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t$$

 Next, using (363) and (373) we infer

$$\nabla_{\dot{\gamma}_0(t)} \Psi^{T_{\gamma_0(t)}M}(\beta_0(t), \cdot)^{-1} P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = -R(T_{\beta_0(t)} \tau_M P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t, \dot{\gamma}_0(t)) \dot{\gamma}_0(t). \quad (378)$$

But, for X_t from (376), $T_{\beta_0(t)} \tau_M P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = Z(t)$, since $T_{\beta_0(t)} \tau_M P_{K_{\beta_0(t)}}^{H_{\beta_0(t)}} X_t = T_{\beta_0(t)} \tau_M X_t = (T\tau_M \circ \omega_M)(\dot{Z}(t)) = \tau_{TM}(\dot{Z}(t)) = Z(t)$. As in our case $V_X = P_{K_X}^{H_X}$, in the left hand side of (378) we can use (366) from Proposition 8 and the usual notation

$$\nabla_{\dot{\gamma}_0(t)} Z(t) = \Psi^{T_{\gamma_0(t)}M}(Z(t), \cdot)^{-1} V_{Z(t)} \dot{Z}(t). \quad (379)$$

In this way we get

$$\nabla_{\dot{\gamma}_0(t)} (\nabla_{\dot{\gamma}_0(t)} Z(t) + T_{\gamma_0(t)}(Z(t), \dot{\gamma}_0(t))) = -R(Z(t), \dot{\gamma}_0(t)) \dot{\gamma}_0(t),$$

or the equation (375) ■

References

1. R. Abraham, J. Robbin, “Transversal Mappings and Flows”, Benjamin, New York, Amsterdam, 1967.
2. S. Kobayashi, K. Nomizu, “Foundations of Differential Geometry”, Interscience Publ. New York, London, vol. I 1963, vol. II 1969.
3. G. Minea, “Entropy conditions for quasilinear first order equations on nonlinear fiber bundles with special emphasis on the equation of 2D flat projective structure. II.”, arXiv:1112.6165v1 [math.AP] 28 Dec. 2011.
4. R. Narasimhan, “Analysis on Real and Complex Manifolds”, Masson, Paris, North Holland, Amsterdam, 1968.
5. R. Palais, “Foundations of Global Non-Linear Analysis”, Benjamin, New York, Amsterdam, 1968.
6. S. Sternberg, “Curvature in Mathematics and Physics”, Dover Publ. 2012.
7. F. Warner, “Foundations of Differentiable Manifolds and Lie Groups”, Springer, 1983.